Here's a word problem that has been attributed to none other than Sir Isaac Newton!

> *Three cows eat all the grass on two acres of land, together with all the grass that grows there in two weeks. Two cows eat all the grass on two acres of land, together with all the grass that grows there in four weeks. How many cows, then, will eat all the grass on six acres of land together with all the grass that grows there in the six weeks?*

Let's solve the problem.

Assume that the quantity of grass on each acre is the same when the cows begin to graze, that the rate of growth is uniform during the time of grazing, and that the cows eat the same amount of grass each week.

We'll use *c* to represent the number of cows that will eat all the grass on six acres of land together with all the grass that grows there in the six weeks.

Also, we'll represent the amount of grass eaten by a cow in one week, the amount of grass already on each acre before the grazing begins and the amount of grass that will grow on one acre in a week as:

C = the amount of grass eaten by one cow in one week.
G = the amount of grass that is on one acre before the grazing begins.
E = the amount of grass that grows on one acre in one week.

Using these constants, we can re-express the given information algebraically.

If "three cows eat in two weeks all the grass on two acres of land, together with all the grass that grows there in the two weeks", we can write: $3\times2\times C=2\times G+2\times2\times E$:

Looking more closely:
$3\times2\times C$ …"three cows eat in two weeks"
= (3 cows) x (2 weeks) x (the amount of grass eaten by one cow in one week).
$2\times G$ …...." all the grass on two acres of land"
= the amount of grass on the two acres before the grazing began.
$2\times2\times E$ …"the grass that grows there in the two weeks"
= (2 acres) x (2 weeks) x (the amount of amount of grass that grows on one acre in one week).

If "two cows eat in four weeks all the grass on two acres of land, together with all the grass that grows there in the four weeks" we can write: $2\times4\times C=2\times G+2\times4\times E$

We now have two equations to work with: (1) $6C=2G+4E$ and (2) $8C=2G+8E$

We can't hope to actually solve these equations, but we can sort out some relationships.

By simplifying (1) we get (3) $3C=G+2E$ and by simplifying (2) we get (4) $4C=G+4E$. Eliminating *G*, we get $C=2E$, which leads us to conclude that $G=4E$. These will be very useful.

Our problem is to find out how many cows will be needed to clear six acres in six weeks. In other words, solve the following equation for *c*.

$$c\times6\times C=6\times G+6\times6\times E\,.$$

Using $C = 2E$ and $G = 4E$, we can simplify this last equation to get: $c \times 6 \times 2E = 6 \times 4E + 6 \times 6 \times E$ which is the same as $12cE = 60E$.

If we divide both sides by 12*E*, we get: $c = 5$. It will take five cows six weeks to clear six acres.

This answers the question that was posed originally. But, surely there must be more that we can say. After all this is supposed to come from Isaac Newton. I'd have thought that there would be more!

We've seen how two cows, three cows and now five cows can clear a number of acres in so many weeks, but could four cows accomplish this and how long would they need? How about six cows? Seven? How about a single cow? Can we answer this question for *N* cows?

This looks like it might be a complicated investigation.

We'll use lower case letters to represent variables and upper case letters to represent the constants.

a = the number of acres
w = the number of weeks
c = the number of cows
C = the amount of grass eaten by one cow in one week.
G = the amount of grass that is on one acre before the grazing begins.
E = the amount of grass that grows on one acre in one week.

We are trying to understand the general case of cows, acres and weeks. Let's proceed by examining what happens when we hold one of the three variables fixed.

Generally, we will assume that the original conditions are to be met. In other words, three cows will eat all the grass on two acres of land in two weeks, and two cows eat all the grass on two acres of in four weeks.

Using these new variables and constants with these same conditions, we have:

(1) $cwC = aG + awE$, where $C = 2E$ and $G = 4E$. (1) becomes $2cw = 4a + wa$.

By isolating each of the variables in this equation, we get three very interesting equations:

2) $c = \frac{a(4+w)}{2w}$, 3) $a = \frac{2wc}{4+w}$, 4) $w = \frac{4a}{2c-a}$.

What happens if we "fix" *w*?

If we assign different values to *w*, we can find a relationship between *a* and *c*.

If $w = 1$, we can see that $w = \frac{4a}{2c-a}$ becomes $1 = \frac{4a}{2c-a}$ and $4a = 2c - a$. This means that $\frac{a}{2} = \frac{c}{5} = t$. Thus $(a, w, c) = (2t, 1, 5t)$, where $t \in I^+$.

For example: $(a, w, c) = (2, 1, 5)$, in which case five cows will clear two acres in one week.
Also, we could have $(a, w, c) = (4, 1, 10)$, in which case ten cows will clear four acres in one week.

If $w=2$, then $4a=2(2c-a)$. This means that $\frac{a}{2}=\frac{c}{3}=t$. Thus $(a,w,c)=(2t,2,3t)$, where $t\in I^{+}$.

For example: $(a,w,c)=(2,2,3)$, with which three cows will clear two acres in two weeks. As we have seen. Also, we could have $(a,w,c)=(4,2,6)$, with which six cows will clear four acres in two weeks.

If $w=3$, then $4a=3(2c-a)$. This means that $\frac{a}{6}=\frac{c}{7}=t$. Thus $(a,w,c)=(6t,2,7t)$, where $t\in I^{+}$.

If $w=k$, then $4a=k(2c-a)$. This means that $\frac{a}{2k}=\frac{c}{4+k}=t$. Thus $(a,w,c)=(2kt,k,(4+k)t)$, where $t\in I^{+}$.

Following in this vein, what happens if $w=a$?

If $w=a$, then $4a=a(2c-a)$. This means that $c=\frac{a+4}{2}$. Thus $(a,w,c)=(2t,2t,t+2)$, where $t\in I^{+}$. This result confirms our solution to the original problem. i.e. if $a=6$, $w=6$, then $c=5$.

How many acres of grass will be cleared, if we let $w=c$, or, if we let the number of weeks be the same as the number of cows. Using the formula $a=\frac{2wc}{4+w}$, and replacing *c* with *w*, we get $a=\frac{2w^2}{4+w}$ or $2w^2-aw-4a=0$.

There are a couple of ways that we can examine this equation while looking for possible values of *a*.

First, we'll examine what happens if we treat this as a quadratic equation in *w*. Using the quadratic formula we see that:

$w=\frac{a\pm\sqrt{a^2-4(2)(-4a)}}{4}$ or $w=\frac{a\pm\sqrt{a^2+32a}}{4}$, and if we want *w* to be a whole number, then a^2+32a has to be a perfect square. We can let $a^2+32a=k^2$ for some whole number *k*. This leads us to $a^2+32a-k^2=0$, another quadratic equation. Using the quadratic formula a second time gives us $a=-16\pm\sqrt{16^2+k^2}$, but this time, if 16^2+k^2 is to be a perfect square, we will be working with a Pythagorean triple and we will have a lot to say about that!

Let $16^2+k^2=M^2$.

Since all Pythagorean triples can be generated using the identity $(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$, we can say that either $m^2-n^2=16$ or $2mn=16$.

If $m^2-n^2=16$, then $m=5$ and $n=3$ is the only possible solution. Thus $M^2=34^2$ and $a=-16\pm\sqrt{34^2}$ or $a=18$.

If $2mn=16$ or $mn=8$, we have four cases to examine: $(m,n)\in((1,8),(2,4),(4,2),(8,1))$. We see that $M^2=20^2$ or $M^2=65^2$, which, in turn yields $a=-16\pm\sqrt{20^2}$ or $a=-16\pm\sqrt{65^2}$ or $a=4$ or $a=49$.

So, we find that if we insist that the number of weeks is the same as the number of cows, there are only three possible values for the number of acres: 4, 18 and 49.

There is another way that we can examine the equation $2w^2 - aw - 4a = 0$. Instead of viewing this as a quadratic, let's isolate the *a* and examine the resulting rational expression. We get $a = \frac{2w^2}{w+4}$ (which we've seen before!). Using long division, this can be re-expressed to look like this: $a = 2w - 8 + \frac{32}{w+4}$. Now, if we want *a* and *w* to be whole numbers, all we have to do is find the values of w that will make $\frac{32}{w+4}$ a whole number.

This is pretty easy. We get $w = 4$, $w = 12$ or $w = 28$. These lead to $a = 4$, $a = 18$ and $a = 49$, which are the same values that we found above.

This second solution is obviously a lot shorter than the first ... maybe even more elegant. However, it is nice to be able to approach problems using various strategies. The first solution required some clever algebraic (geometric!) footwork and utilized a rather unusual aspect of the quadratic formula (insisting that the discriminant be a perfect square). It also required a rather esoteric piece of information about Pythagorean triples. Personally, I find this use of multiple mathematical tools quite satisfying.

This problem has been attributed to Sir Isaac Newton and appears in a number of articles, books and on a number of websites. There are many very interesting discussions about the problem, which are worthwhile examining.

However, I have always had my doubts about Sir Isaac actually being all that interested in this problem.

I found a reference to "Newton's Problems of the Fields and Cows" in a book titled 100 Great Problems of Elementary Mathematics: Their History and Solution." by Heinrich Dörrie - a Dover publication. Apparently this book had been sitting on one of my bookshelves for quite a long time. I originally bought it for $3.25! This is how the problem appears there:

In Newton's *Arithmetica Universalis* (1707) the following interesting problem is posed:

> *a cows graze b acres bare in c weeks,*
> *a' cows graze b' acres bare in c' weeks,*
> *a'' cows graze b'' acres bare in c'' weeks;*

what relation exists between the nine magnitudes a to c'' ?

Dörrie provides a neat solution which is surprisingly similar to the one described above, but with a huge difference. The very nature of the problem has changed. Apparently, Newton is looking for a general condition rather than a solution to a specific case.

Using Newton's notation with our strategy we can quickly derive the following system of equations:

C = the amount of grass eaten by one cow in one week.
G = the amount of grass that is on one acre (field)? before the grazing begins.
E = the amount of grass that grows on one acre (field?) in one week.

Using the previous approach to creating our equations we get:

$$bG + bcE - acC = 0$$
$$b'G + b'c'E - a'c'C = 0$$
$$b''G + b''c''E - a''c''C = 0$$

We have a number of ways to proceed with this. I personally like the following.

Let's treat this system as the equations of three planes. We will consider (G,E,C) as our variables and *the nine magnitudes a to c''* as constants. $\overrightarrow{v_1}=(b,bc,-ac)$, $\overrightarrow{v_1}=(b',b'c',-a'c')$ and $\overrightarrow{v_1}=(b'',b''c'',-a''c'')$ are the normals to the three planes.

Now, it gets interesting. This is a homogeneous system so we know that the three planes have to intersect at the origin. But if this were the only solution, this would mean that $(G,E,C)=(0,0,0)$ which would preclude any growth or consumption of the grass (or interest in the problem!). So, if there is to be a non-zero solution to this system the three normals have to be dependent vectors. Incidentally, a second solution would imply an infinity of solutions!

If the normals are to be dependent then we know that $\overrightarrow{v_1}\cdot\overrightarrow{v_2}\times\overrightarrow{v_3}=0$. Alternatively, we could write this using deteminants:

$$\begin{vmatrix} b & bc & -ac \\ b' & b'c' & -a'c' \\ b'' & b''c'' & -a''c'' \end{vmatrix}=0$$

Expressing the original problem in these terms certainly gives us our answer quickly.

Three cows (a=3) eat all the grass on two acres (b=2) of land, together with all the grass that grows there in two weeks (c=2). Two cows (a'=2) eat all the grass on two acres (b'=2) of land, together with all the grass that grows there in four weeks (c'=4). How many cows (a''=?), then, will eat all the grass on six acres (b''=6) of land together with all the grass that grows there in the six weeks (c''=6)?

With the appropriate substitutions, we get:

$$\begin{vmatrix} 2 & 2\times 2 & -3\times 2 \\ 2 & 2\times 4 & -2\times 4 \\ 6 & 6\times 6 & -a''\times 6 \end{vmatrix}=0$$

Expanding this and solving, we get:

$$2\Big(8\times(-6a'')-\big(36\times(-8)\big)\Big)-4\Big(2\times(-a''\times 6)-6\times(-8)\Big)-6\big(2\times 36-6\times 8\big)=0$$

$$\therefore -96a''+576+48a''-192-144=0$$

$$\therefore a''=5$$

So, 5 cows will clear 6 acres in 6 weeks: as we've seen before.

As you’ve probably guessed by now, I became a bit obsessed with this problem and as you might expect, couldn’t let it go here. I just had to have a look at the original manuscript. There are facsimile copies of the original 1707 book *Arithmetica Universalis* available on the web – I’ve included a link to a PDF copy of the English version which was published in 1720.

Newton never actually published this book. In fact he wouldn't allow his name to appear on the cover. Apparently the book is a set of lecture notes that he wrote between 1673 and 1683. It's not known if he ever delivered these lectures.

I have transcribed the pertinent section from the English version.

Problem XI. If the number of Oxen *a* eat up the Meadow *b* in time *c*; and the Number of Oxen *d* eat up as good a Piece of Pasture *e* in time *f*, and the Grass grows uniformly; to find how many Oxen will eat up the like Pasture *g* in time *h*.

If the Oxen *a* in the Time *c* eat up the Pasture *b*; then by Proportion, the Oxen $\frac{e}{b}a$ in Time *c*, or the Oxen $\frac{ec}{bf}a$ in the Time f, or the Oxen $\frac{ec}{bh}a$ in the Time *h* will eat up the Pasture *e*; supposing the Grass did not grow [at all] after the Time *c*. But since, by reason of the Growth of the Grass, all of the Oxen d in the Time *f* can eat up only the Meadow *e*, therefore that Growth of the Grass in the meadow *e*, in the Time $f-c$, will be so much as alone would be sufficient to feed the Oxen $d-\frac{eca}{bf}$ [in] the Time *f*, that is as much as would suffice to feed the Oxen $\frac{df}{h}-\frac{eca}{bh}$ in the Time *h*. And in the Time $h-c$, by Proportion, so much would be the Growth of the Grass as would be sufficient to feed the Oxen $\frac{h-c}{f-c}$ into $\frac{df}{h}-\frac{eca}{bh}$ or $\frac{bdfh-ecah-bdcf+aecc}{bfh-bch}$. Add this Increment to the Oxen $\frac{aec}{bh}$, and there will come out $\frac{bdfh-ecah-bdcf+ecfa}{bfh-bch}$ - the Number of Oxen which the Pasture *e* will suffice to feed during the Time *h*. And so by [in] Proportion the Meadow *g* will suffice to feed the Oxen $\frac{gbdfh-ecagh-bdcgf+ecfga}{befh-bceh}$ during the same Time *h*.

Example. If 12 Oxen eat up $3\frac{1}{3}$ Acres of Pasture in 4 Weeks, and 21 Oxen eat up 10 Acres of like Pasture in 9 Weeks; to find how many Oxen will eat up 24 [*there is a typo in the manuscript! It originally read 36*] Acres in 18 Weeks? Answer 36; for the Number will be found by substituting in $\frac{gbdfh-ecagh-bdcgf+ecfga}{befh-bceh}$ the Numbers 12, $3\frac{1}{3}$, 4, 21, 10, 9, 36 and 18 for the Letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h* respectively; but the Solution, perhaps, will be no less expedite, if it be brought out from the first Principles, in Form of the precedent literal Solution. As if 12 Oxen in 4 Weeks eat up $3\frac{1}{3}$ Acres, then by Proportion 36 Oxen in 4 Weeks, or 16 Oxen in 9 Weeks, or 8 Oxen in 18 Weeks, will eat up 10 Acres, on

> Supposition that the Grass did not grow. But since by reason of the Growth of the Grass 21 Oxen in 9 Weeks can eat up only 10 Acres, that Growth of the Grass in 10 Acres for the last 5 Weeks will be as much as would be sufficient to feed 5 Oxen, that is the Excess of 21 above 16 for 9 Weeks, or, what is the same Thing, to feed (??? – *can't read the copy*!) Oxen for 18 Weeks. And in 14 Weeks (the Excess of 18 above the first 4) the Increase of the Grass, by Analogy, will be such, as to be sufficient to feed 7 Oxen for 18 Weeks: Add there 7 Oxen, which to the Growth of the Grass alone would suffice to feed, to the 8, which the Grass without Growth after 4 Weeks would feed, and the Sum will be 15 Oxen. And, lastly, if 10 Acres suffice to feed 15 Oxen for 18 Weeks, then, in Proportion, 24 Acres would suffice 36 Oxen for the same Time.

Now, I found this absolutely fascinating because I didn't have a clue what he was talking about. I understood the words but how they wove together was a mystery! I couldn't imagine sitting in a lecture hall and understanding any of this! But I like a puzzle and so I sat down and plowed my way through. Here is my reading of the argument.

Let's rephrase the first paragraph of his general treatment.

> *a oxen graze b acres bare in c weeks,*
> *d oxen graze e acres bare in f weeks,*
> *x oxen graze g acres bare in h weeks;*

Using ratios we can alter the first statement so that it relates to *e* acres. We can work with ratios as long as we are dealing with the number of oxen and the number of acres. To illustrate, if *a* oxen clear *b* acres in *c* weeks, then, if we increase (decrease?) the number of acres, then we should be able to increase the number of oxen required to clear those acres, proportionately. If you double the number of acres, it stands to reason that you'd need to double the number of oxen, if you want the job done in the same number of weeks.

So, if we are to increase (decrease?) *b* to *e*, you need to multiply *b* by $\frac{e}{b}$. But this means that you also have to multiply *a* by $\frac{e}{b}$. The "altered" first statement now reads "$\frac{ae}{b}$ oxen graze *e* acres in *c* weeks".

If we are going to adjust the number of weeks we have to be aware of the fact that new grass is growing. This prevents us from using simple ratios. However, Newton deals with this issue in a very clever way.

For now, we will ignore the new growth and return to it a bit later (just as Sir Isaac did!)

It is reasonable to say that to increase (decrease) the number of weeks from *c* to *f*, we can simply multiply *c* by $\frac{f}{c}$. If we keep the number of acres the same, we will have to decrease (increase) the number of oxen proportionately. So, to calculate the number of oxen required to clear e acres in f weeks, we just multiply $\frac{ae}{b}$ by $\frac{c}{f}$ (the *reciprocal* of $\frac{f}{c}$). Therefore $\frac{ace}{bf}$ oxen graze *e* acres in *f* weeks.

Similarly $\frac{ace}{bh}$ $\left(i.e. \frac{ae}{b}\times\frac{c}{h}\right)$ oxen will graze e acres in h weeks. (ignoring the new growth during the time of $h-c$ weeks!!!!)

Now, we know that "*d oxen graze e acres bare in f weeks*", and this does take into account all of the new growth. This tells us that the difference, $d-\frac{ace}{bf}$, must be the number of oxen required to eat the new growth that occurs on e acres in $f-c$ weeks. (Think about this, it deserves a bit of thought... rumination even???)

The $d-\frac{ace}{bf}$ oxen eat the new grass that grows in $f-c$ weeks, but these oxen would have been around and eating this grass for the full f weeks.

We can ramp this argument up to consider what happen in h weeks. If $d-\frac{ace}{bf}$ oxen take f weeks to eat the new grass that grows in $f-c$ weeks then $\frac{f}{h}\left(d-\frac{ace}{bf}\right)$ oxen will take h weeks to eat the same amount of grass.

Note: $\frac{f}{h}\left(d-\frac{ace}{bf}\right)=\frac{bdf-ace}{bh}$.

Presumably the amount of grass produced in $f-c$ weeks will be proportional to that produced in $h-c$ weeks, as will the number of oxen required to eat it, as long as they have the same amount of time to do this - i.e. h weeks. So, if we let p represent the number of oxen needed to eat the new growth on e acres during the period $h-c$ weeks, we will have:

$$\frac{p}{\frac{bdf-ace}{bh}}=\frac{h-c}{f-c}$$

$$\therefore \frac{pbh}{bdf-ace}=\frac{h-c}{f-c}$$

$$\therefore p=\frac{(bdf-ace)(h-c)}{(bh)(f-c)}$$

Now we already know that $\frac{ace}{bh}$ oxen will clear e acres in h weeks if we ignore the new growth after c weeks. Therefore we need to add this number to p in order to find the number of oxen needed to clear all of the grass (new growth and old) that will appear in h weeks.

$$\begin{aligned}\therefore \frac{ace}{bh}+p&=\frac{ace}{bh}+\frac{(bdf-ace)(h-c)}{(bh)(f-c)}\\&=\frac{bdhf-aceh-bcdf+acef}{bfh-bhc}\end{aligned}$$

This is the total number of oxen required to clear all of the grass (old and new!) on e acres in h weeks. To adjust this to cover g acres in h weeks, we must multiply by $\frac{g}{e}$.

Our final answer (which happens to be the same as Sir Isaac's!) is:

$$x = \frac{bdfgh - acegh - bcdfg + acefg}{befh - bceh}.$$

To be perfectly honest, I don't know if my explanation is any easier to follow than the original. I've had to struggle with this in both trying to understand what Newton was saying and trying to write out my own thoughts. Perhaps you might like to give it a go!

But there are a couple of observations that I would like to throw out.

I think that it is very interesting how the more modern approach (mine??) seems to move the problem from the literal to the algebraic as soon as possible. Once the problem has been "algebratized" we can go to town with all of our heavy equipment and knock out a quick solution. I think that it's cool that we can link the original problem to so many seemingly unrelated concepts... e.g. the Quadratic formula, Pythagorean triples and dependent vectors. This was accomplished primarily by the introduction of a new constant - the amount of grass that grows on one acre in one week. (It's odd that Sir Isaac didn't think of this!)

On the other hand, Newton never really left the "original realm" of the problem. The problem was stated literally and his solution proceeded in the same mode. All of his proportional reasoning was based on the actual real-world conditions of the original problem. To think that this may have been presented in a lecture setting. I shudder at the thought of being a student in that room.