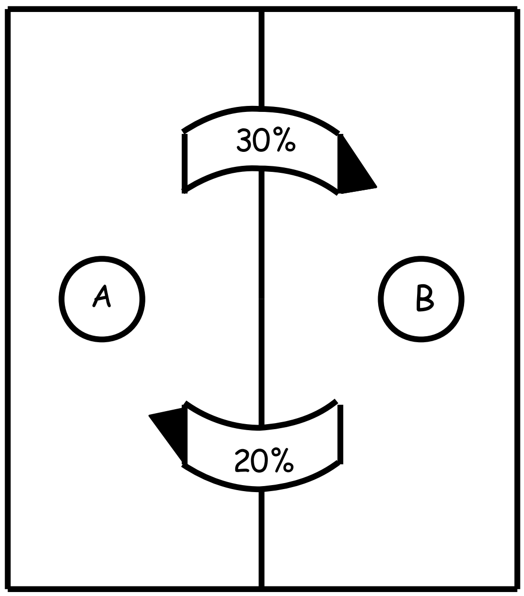
"Migrating" Cows

In the diagram below, the percentages and arrows on each bridge indicate the **daily rate of migration** of cows from one field to another.



If you are to obey the "cow-friendly" rules, what is the smallest total number of cows that would make this puzzle possible?

Is it possible that the number of cows in **each** field doesn't change with migration? What is the smallest total number of cows that would make this happen?

Starting with A = 90 and B = 10, create a spreadsheet to investigate what happens to the distribution of cows after several days of migration.

Repeat the above experiment with different initial distributions of cows. What do you notice?
Can you draw any conclusions?

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The smallest cow-friendly values for A and B have to be 10 and 5. The smallest total number of cows is 15.
If the number of cows in A and B are to remain unchanged, then 30% of A will have to equal 20% of B. In this way there are just as many cows leaving field A as are coming in. The same is true for field B. Thus: $.3A = .2B$.
Therefore $A = \frac{2}{3}B$. The smallest cow-friendly values for A and B have to be 10 and 5, giving a total of 15.
Here are two spreadsheets showing what happens with different initial distributions of cows. You should notice that the ratio A:B ends up the same in both cases.

Ainit	90.0000		flow A->B		0.3
Binit	10.0000		flow B->A		0.2

n	An	Bn		A/B
0	90.0000	10.0000		9.000000
1	65.0000	35.0000		1.857143
2	52.5000	47.5000		1.105263
3	46.2500	53.7500		0.860465
4	43.1250	56.8750		0.758242
5	41.5625	58.4375		0.711230
6	40.7813	59.2188		0.688654
7	40.3906	59.6094		0.677588
8	40.1953	59.8047		0.672110
9	40.0977	59.9023		0.669384
10	40.0488	59.9512		0.668024
11	40.0244	59.9756		0.667345
12	40.0122	59.9878		0.667006
13	40.0061	59.9939		0.666836
14	40.0031	59.9969		0.666751
15	40.0015	59.9985		0.666709
16	40.0008	59.9992		0.666688
17	40.0004	59.9996		0.666677
18	40.0002	59.9998		0.666672
19	40.0001	59.9999		0.666669
20	40.0000	60.0000		0.666668
21	40.0000	60.0000		0.666667
22	40.0000	60.0000		0.666667
23	40.0000	60.0000		0.666667
24	40.0000	60.0000		0.666667
25	40.0000	60.0000		0.666667

Ainit	20.0000		flow A->B		0.3
Binit	50.0000		flow B->A		0.2

n	An	Bn		A/B
0	20.0000	50.0000		0.400000
1	24.0000	46.0000		0.521739
2	26.0000	44.0000		0.590909
3	27.0000	43.0000		0.627907
4	27.5000	42.5000		0.647059
5	27.7500	42.2500		0.656805
6	27.8750	42.1250		0.661721
7	27.9375	42.0625		0.664190
8	27.9688	42.0313		0.665428
9	27.9844	42.0156		0.666047
10	27.9922	42.0078		0.666357
11	27.9961	42.0039		0.666512
12	27.9980	42.0020		0.666589
13	27.9990	42.0010		0.666628
14	27.9995	42.0005		0.666647
15	27.9998	42.0002		0.666657
16	27.9999	42.0001		0.666662
17	27.9999	42.0001		0.666664
18	28.0000	42.0000		0.666665
19	28.0000	42.0000		0.666666
20	28.0000	42.0000		0.666666
21	28.0000	42.0000		0.666667
22	28.0000	42.0000		0.666667
23	28.0000	42.0000		0.666667
24	28.0000	42.0000		0.666667
25	28.0000	42.0000		0.666667

Notice that the total number of cows never changes. This means that $A+B=k$. If we represent this on a graph we see that the initial point $P_0(A,B)$- where A and B are the numbers of cows in their respective fields – as well as all of the subsequent points (A',B') will all lie on the line $A+B=k$.

The "fixed" point of this migration (A_f,B_f) (i.e. the point that doesn't change) must also be on this line. But we know that this "fixed" point has $A_f:B_f=2:3$. This means that (A_f,B_f) must also lie on the line $B=\frac{3}{2}A$.

We can easily show that $(A_f,B_f)=\left(\frac{2}{5}(A+B),\frac{3}{5}(A+B)\right)$. Let's call this point *W*.

Now, the interesting question that comes up is this – Is W an "attracting" fixed point? In other words, will any initial point (A,B) such that $A+B=k$, ultimately migrate to the point *W*?

On the right is a spreadsheet that illustrates this idea. $A+B=150$. $A_{n+1}=.7A_n+.2B_n$ and $B_{n+1}=.3A_n+.8B_n$. You can see that $(A_0,B_0)=(100,50)$ ultimately migrates to the "fixed point" $(A_f,B_f)=(60,90)$.

iteration	A	B
0	100.0000	50.0000
1	80.0000	70.0000
2	70.0000	80.0000
3	65.0000	85.0000
4	62.5000	87.5000
5	61.2500	88.7500
6	60.6250	89.3750
7	60.3125	89.6875
8	60.1563	89.8438
9	60.0781	89.9219
10	60.0391	89.9609
11	60.0195	89.9805
12	60.0098	89.9902
13	60.0049	89.9951
14	60.0024	89.9976
15	60.0012	89.9988
16	60.0006	89.9994
17	60.0003	89.9997
18	60.0002	89.9998
19	60.0001	89.9999
20	60.0000	90.0000

If you were to change the initial values for (A,B) with $A+B=150$, you will find that the ultimate outcome is always the same. Thus it looks as if $W(60,90)$ is indeed an "attracting" fixed point.

Can we prove this? Can we give an accurate description of just how this happens?

Let's compare $|P_0P_1|$ to $|P_0W|$. In other words we'll calculate the distance between P_0 and P_1 and the distance between P_0 and W and look for a relationship.

With $P_0=(A,B)$ and $W=\left(\frac{2}{5}(A+B),\frac{3}{5}(A+B)\right)$, we see that:

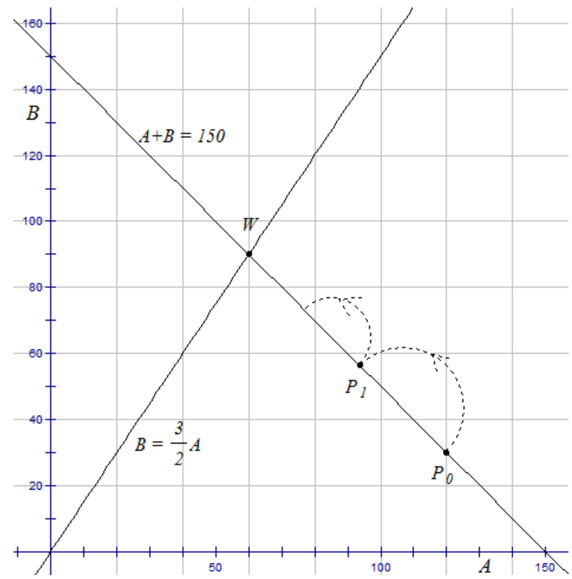
$$\begin{aligned}|P_0W|&=\sqrt{\left(A-\frac{2}{5}(A+B)\right)^2+\left(B-\frac{3}{5}(A+B)\right)^2}\\&=\sqrt{\left(\frac{3}{5}A-\frac{2}{5}B\right)^2+\left(\frac{2}{5}B-\frac{3}{5}A\right)^2}\\&=\frac{\sqrt{2}}{5}|3A-2B|\end{aligned}$$

With $P_0=(A,B)$ and $P_1=(.7A+.2B,.3A+.8B)$, we see that:

$$\begin{aligned}|P_0P_1|&=\sqrt{\left(A-(.7A+.2B)\right)^2+\left(B-(.3A+.8B)\right)^2}\\&=\sqrt{(.3A-.2B)^2+(.2B-.3A)^2}\\&=\frac{\sqrt{2}}{10}|3A-2B|\end{aligned}$$

This shows that $\frac{P_0P_1}{P_0W} = \frac{1}{2}$.

In other words, with each iteration, the point P_{i+1} will be half as close to W as P_i. The graph on the right illustrates this.



Notice that P_1 is the midpoint of P_0W. This brings to mind a "Zeno-like" cow hopping towards the point W with ever decreasing steps. This illustrates that W is indeed an attracting fixed point.

Here is a very different approach to this problem.

Let $P_0(A,B)$ and $P_1(a,b)$ be the initial and second distributions of cows in the respective fields. We know that $a = .7A + .2B$ and $b = .3A + .8B$.

Using some ideas from Linear Algebra (eigen values and eigen vectors), we ask the question "is it possible that $a = \lambda A$ and $b = \lambda B$, for some value of λ?". We already know that λ can be 1 - this corresponds to the "fixed" point, but is there some other possible value?

On the surface, this seems like a pretty silly question, because if λ were some value other than 1, then the total number of cows would have to change. This means that we are either going to lose some cows or create some new ones and this seems entirely unreasonable. But I would like you to bear with me for just a little while.

Let's investigate the equations $a = .7A + .2B = \lambda A$ and $b = .3A + .8B = \lambda B$.

We have $7A + 2B = 10\lambda A$ or $2B = (10\lambda - 7)A$ (i) and $3A + 8B = 10\lambda B$ or $A = \frac{10\lambda - 8}{3}B$ (ii).

Combining (i) and (ii) yeilds $2B = (10\lambda - 7)\frac{(10\lambda - 8)}{3}B$.

This simplifies to $0 = 2\lambda^2 - 3\lambda + 1$, which has the roots $\lambda = 1$ and $\lambda = \frac{1}{2}$.

If $\lambda = 1$, then $7A + 2B = 10A$, which gives us $2B = 3A$. e.g. $(A,B) = (2,3)$ - which we already know gives a fixed point.

If $\lambda = \frac{1}{2}$, then $7A + 2B = 5A$, which gives us $B = -A$. e.g. $(A,B) = (1,-1)$ - this is going to need some explaining! But for now, let's just follow the current line of reasoning.

What this all means is the following. If the distribution of the cows in the two fields is $(A,B) = (2,3)$, then $A_n = A(1)^n$ and $B_n = B(1)^n$. (thus $(A_n, B_n) = (A,B)$, the numbers of cows in the two fields don't change).

If the distribution of the cows in the two fields is $(A,B) = (1,-1)$, then $A_n = A\left(\frac{1}{2}\right)^n$ and $B_n = B\left(\frac{1}{2}\right)^n$. In other words, with each iteration, the number of cows in each field is reduced by half! Again I ask for your patience.

Now here comes the interesting part. $(A,B)=(2,3)$ and $(A,B)=(1,-1)$ are actually "independent vectors". This means that no matter what the initial distribution of cows is, (A,B) can be written as a linear combination of the two vectors $(A,B)=(2,3)$ and $(A,B)=(1,-1)$. If we do this, we will be able to write explicit functions for A_n and B_n in terms of *n*.

Let's consider a specific example before we tackle the general case.

Let $P_0(A,B)=(50,100)$. (note $A+B=150$, as above)

We require that $(50,100)=p(2,3)+q(1,-1)$. This gives $p=30$ and $q=-10$. So if we divide the cows into two groups, distributed as $(60,90)$ and $(-10,10)$, the first group will behave according to the rules $A_n=A(1)^n$ and $B_n=B(1)^n$ while the second group will behave according to the rules $A_n=A\left(\frac{1}{2}\right)^n$ and $B_n=B\left(\frac{1}{2}\right)^n$.

Taken together we see that $A_n=60(1)^n-10\left(\frac{1}{2}\right)^n$ and $B_n=90(1)^n+10\left(\frac{1}{2}\right)^n$.

If $n=0$, we have $P_0(A,B)=(50,100)$, as expected and as $n\to\infty$ we see that $W(A_f,B_f)=(60,90)$. We have seen this before, but now we have more than a spreadsheet to go on!

The spreadsheet on the right was created using the iterative formulae $A_{n+1}=.7A_n+.2B_n$ and $B_{n+1}=.3A_n+.8B_n$ for the columns A and B and the explicit functions, $A_n=60(1)^n-10\left(\frac{1}{2}\right)^n$ and $B_n=90(1)^n+10\left(\frac{1}{2}\right)^n$ for the columns An and Bn.

iteration	A	B	An	Bn
0	50.0000	100.0000	50.0000	100.0000
1	55.0000	95.0000	55.0000	95.0000
2	57.5000	92.5000	57.5000	92.5000
3	58.7500	91.2500	58.7500	91.2500
4	59.3750	90.6250	59.3750	90.6250
5	59.6875	90.3125	59.6875	90.3125
6	59.8438	90.1563	59.8438	90.1563
7	59.9219	90.0781	59.9219	90.0781
8	59.9609	90.0391	59.9609	90.0391
9	59.9805	90.0195	59.9805	90.0195
10	59.9902	90.0098	59.9902	90.0098
11	59.9951	90.0049	59.9951	90.0049
12	59.9976	90.0024	59.9976	90.0024
13	59.9988	90.0012	59.9988	90.0012
14	59.9994	90.0006	59.9994	90.0006
15	59.9997	90.0003	59.9997	90.0003
16	59.9998	90.0002	59.9998	90.0002
17	59.9999	90.0001	59.9999	90.0001
18	60.0000	90.0000	60.0000	90.0000
19	60.0000	90.0000	60.0000	90.0000
20	60.0000	90.0000	60.0000	90.0000

The results appear to be the same!

We can take this a little further. We can prove that these explicit functions are correct using the **method of mathematical induction.**

Proof:
step 1. Show that the proposition is true for the first case. Here the first case occurs when $n=0$.

$A_0=60(1)^0-10\left(\frac{1}{2}\right)^0=50$ and $B_0=90(1)^0+10\left(\frac{1}{2}\right)^0=100$.

step 2. Assume that the propostion is true for $n\le k$ and show that it is also true for $n=k+1$.

Thus we assume that $A_k=60(1)^k-10\left(\frac{1}{2}\right)^k$ and $B_k=90(1)^k+10\left(\frac{1}{2}\right)^k$.

But $A_{k+1} = .7A_k + .2B_k$ and $B_{k+1} = .3A_k + .8B_k$, therefore:

$$A_{k+1} = .7\left(60(1)^k - 10\left(\frac{1}{2}\right)^k\right) + .2\left(90(1)^k + 10\left(\frac{1}{2}\right)^k\right)$$
$$= 42(1)^k - 7\left(\frac{1}{2}\right)^k + 18(1)^k + 2\left(\frac{1}{2}\right)^k$$
$$= 60(1)^k - 5\left(\frac{1}{2}\right)^k$$
$$= 60(1)(1)^k - 10\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^k$$
$$= 60(1)^{k+1} - 10\left(\frac{1}{2}\right)^{k+1}$$

$$B_{k+1} = .3\left(60(1)^k - 10\left(\frac{1}{2}\right)^k\right) + .8\left(90(1)^k + 10\left(\frac{1}{2}\right)^k\right)$$
$$= 18(1)^k - 3\left(\frac{1}{2}\right)^k + 72(1)^k + 8\left(\frac{1}{2}\right)^k$$
$$= 90(1)^k + 5\left(\frac{1}{2}\right)^k$$
$$= 90(1)(1)^k + 10\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^k$$
$$= 90(1)^{k+1} + 10\left(\frac{1}{2}\right)^{k+1}$$

This proves our assertion and is a lot more convincing than just examining a spreadsheet!

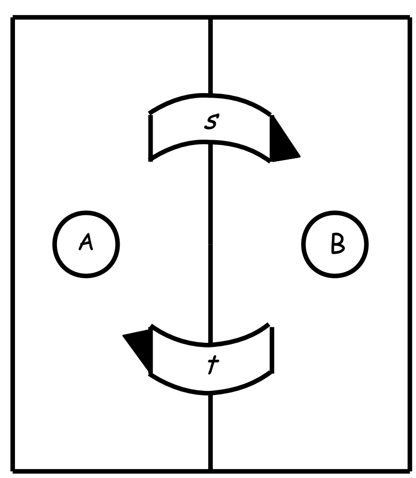
Before discussing the general case, let's examine some of the weird developments that we have witnessed in the recent discussion.

We have broken all the rules. Never mind the introduction of "fractional" cows, we have been dealing with "negative" cows and I don't mean their emotional disposition! The pay off, of course, is that by allowing these preposterous ideas to enter the discussion, we have been able to create a very nice piece of mathematics. Being able to convert the iterative functions into explicit functions has to be seen as a beautiful development.

There is a precedent for this. If you have a look at the wonderful book Ars Magna (or the Rules of Algebra) by Girolamo Cardano, published in 1545, you will see the introduction of another "preposterous" idea - $\sqrt{-1}$ - which helped him to find the three rational roots of a cubic equation. By introducing an imaginary number into his work and following the logic, Cardano expanded the boundaries of know mathematics. Perhaps I can be forgiven for talking about "imaginary" cows.

It should also be noted, that with $(A,B) = (1,-1)$, the total number of cows in this portion of the herd would have to be zero. Thus halving *A* and *B* wouldn't actually change the fact that the complete herd of cows wouldn't be reduced. This is a good thing, if a little weird.

The General Case:



Consider the diagram to the right. s and t are rational numbers which indicate the daily rate of migration in the directions indicated.

Let's develop the explicit functions which give the number of cows in each of the fields with each passing day.

Following the previous argument, we can see that $a = (1-s)A + tB = \lambda A$ and $b = sA + (1-t)B = \lambda B$. This should be enough for you to show that $\lambda = 1$ or $\lambda = 1 - s - t$.

We see that if $\lambda = 1$, then the distribution of the cows in the two fields should be $(A,B) = (t,s)$ and $A_n = A(1)^n$ and $B_n = B(1)^n$.

On the other hand, if $\lambda = 1 - s - t$, then the distribution of the cows in the two fields should be $(A,B) = (1,-1)$ and $A_n = A(1-s-t)^n$ and $B_n = B(1-s-t)^n$.

Again, following the previous argument, we will need to find *p* and *q* such that $(A,B) = p(t,s) + q(1,-1)$.

We will get the following: $p = \frac{A+B}{s+t}$ and $q = \frac{As-Bt}{s+t}$.

So if we divide the cows into two groups, distributed as $p(t,s) = \left(\left(\frac{A+B}{s+t}\right)t, \left(\frac{A+B}{s+t}\right)s\right)$ and $q(1,-1) = \left(\frac{As-Bt}{s+t}, -\frac{As-Bt}{s+t}\right)$, the first group will behave according to the rules $A_n = A(1)^n$ and $B_n = B(1)^n$ while the second group will behave according to the rules $A_n = A(1-s-t)^n$ and $B_n = B(1-s-t)^n$.

Taken together we see that $A_n = \frac{t(A+B)}{s+t}(1)^n + \frac{As-Bt}{s+t}(1-s-t)^n$ and $B_n = \frac{s(A+B)}{s+t}(1)^n - \frac{As-Bt}{s+t}(1-s-t)^n$.