I#1 **Sum:**

q.2: T.R.

$i)\ \ 7 = A + 4 \quad \therefore A = 3$

$ii)\ \ A + 2 = C \quad \therefore C = 5$

$iii)\ B = 4 + 2 \quad \therefore B = 6$

q.3: B.L.

$i)\ \ x = 8 + 7 \quad \therefore x = 15$

$ii)\ \ z = 8 + 9 \quad \therefore z = 17$

$iii)\ \ y = 7 + 9 \quad \therefore y = 16$

q.4: B.R

$i)\ \ 5 + A = 9 \quad \therefore A = 4$

$ii)\ \ 12 = 4 + B \quad \therefore B = 8$

$iii)\ x = 5 + 8 \quad \therefore x = 13$

These questions are fairly straight forward. Students need to remember that the number on the bridge is the "sum" while the numbers in the adjoining fields are "addends".

I#2 **Sum:**

q.1: T.L.

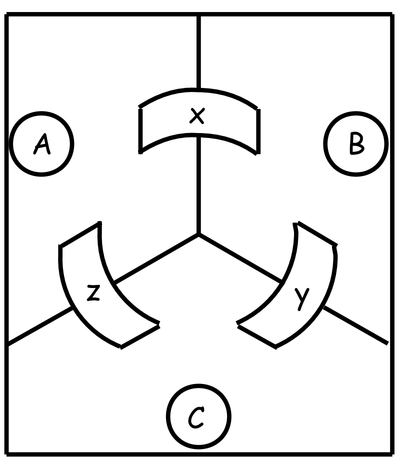
$i)\ \ 15 + A = 33 \quad \therefore A = 18$

$ii)\ \ A + 12 = y \quad \therefore 18 + 12 = y \quad \therefore y = 30$

$iii)\ 15 + 12 = x \quad \therefore x = 27$

Before discussing q.2 TR, you might want to consider the following.
What is the total number of cows in the three fields?
What is the sum of the numbers on the three bridges?
How are these two totals related? Is there a reason for this?

The total of the numbers on the three bridges is twice the total number of cows. This can be explained orally by pointing out that when we add the numbers on the bridges we are actually adding each of the numbers in the fields twice. Algebraically, this looks like this:



$$A + B = x$$
$$B + C = y$$
$$C + A = z$$
$$\therefore (A + B) + (B + C) + (C + A) = x + y + z$$
$$\therefore (A + B + C) + (A + B + C) = x + y + z$$
$$\therefore 2(A + B + C) = x + y + z$$

Also, let $A + B + C = K$, $\therefore x + C = K,\ y + A = K,\ z + B = K$.
Therefore $x + C = y + A = z + B$: which has a nice symmetric ring to it.

q.2: T.R.
We can determine the value of x quickly by noticing that $x = 22 + 31$ or $x = 53$. But determining the other values might be a bit tricky. However, if we follow the ideas explained previously, we know that: $2(A + B + C) = x + y + z$. This means that $2(A + B + C) = 140$ or $A + B + C = 70$. We know that $A + B = 53$ and so $C = 70 - 53$ or $C = 17$. Finding y and z is now pretty easy: $y = 48$ and $z = 39$.

q.3: B.L.
B is obtained by noting that: $10 + B = 21$. Thus $B = 11$.
Notice also that $C + 10 = z$ and $C + 11 = y$. We can combine these two equations to get $2C + 21 = z + y$. Since $x + y = 45$, we have $2C + 21 = 45$. Therefore $C = 12$, $z = 22$ and $y = 23$.

I was once working on this puzzle with a group of grade seven students. After a rather long period of silence, one of the girls pointed out that it was "obvious" that y and z had to be consecutive numbers and that $z = 22$ and $y = 23$. After thinking about this for a few minutes, I finally realized that this was "obvious"!!!!!!

q.4: B.R.
Since $2(A + B + C) = x + y + z$, with $A + B + C = 38$, $x = 25$ and $z = 28$, then we have $2(38) = 25 + y + 28$. Therefore $y = 23$. If $A + B = 25$, then $C = 13$. If $A + C = 28$, then $B = 10$. If $B + C = 23$, then $A = 15$.

A question like this can also be approached in a more naive fashion. The method of "guess and check" can be used to good effect. Let's set B equal to 5 (or any whole number less than 25). Thus A would be 20 and C would have to be 8. This makes $y = 15$. Check: $A + B + C = 20 + 5 + 8$. Thus $A + B + C = 33$: which is too small. If we were to chose a larger value for B would this increase or decrease the value of C? By how much? A little thinking will guide you to the correct solution.

I#3 **Sum:**
The numbers on each bridge is the sum of the numbers in the adjoining fields.

There are several ways to approach these two problems. Here's one way to approach the puzzle on the left. We know from above that $2(A + B + C) = x + y + z$. This means that $2(A + B + C) = 54 + 56 + 55$ or $2(A + B + C) = 165$. Which means that $A + B + C = \frac{165}{2}$. This can happen only if we allow parts of cows to inhabit our fields. This isn't cow-friendly, so we have to say that the problem is not solvable!

The other puzzle can be approached in a similar fashion. Here we find that $2(A + B + C) = 17 + 18 + 19$. This means that $A + B + C = 27$ - which is cow-friendly. In fact we can easily determine the number of cows in each field. Noting that $A + B = 18$, we get $C = 9$. Since $B + C = 19$, A must be 8. Finally, since $A + C = 17$, $B = 10$.

In the second puzzle, the total of the numbers on the bridges is an even number. In the first puzzle, this is an odd number. Since we do not allow fractions in our answers, we have to conclude that the first puzzle cannot be solved using cow-friendly numbers.

Comment: if we were to drop our "cow-friendly" restriction, both puzzles could be solved.

These last few questions are somewhat difficult. Since they require more algebraic reasoning, students may need some guidance in navigating their way. Students will need to connect various pieces of the puzzle and pay particular attention to the clarity of their written solutions. There is a gradual accumulation of results being used to create the solutions shown here. Hopefully, students will appreciate how we can develop problem solving skills by acquiring and utilizing general results. This is a critical feature of mathematics and mathematical thinking.

I#4 **Sum:**

q1. TL

i) $B+10=25 \therefore B=15$

ii) $A+B=24 \therefore A+15=24 \therefore A=9$

iii) $A+D=18 \therefore 9+D=18 \therefore D=9$

iv) $D+10=x \therefore 9+10=x \therefore x=19$

q2. TR

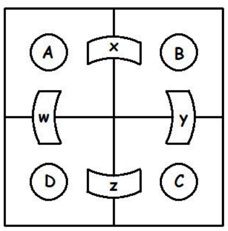
i) $12+C=33 \therefore C=21$

ii) $D+C=25 \therefore D+21=25 \therefore D=4$

iii) $A+D=21 \therefore A+4=21 \therefore A=17$

iv) $A+12=x \therefore 17+12=x \therefore x=29$

If we take a closer look at these questions, we will find some nice general results that we will find quite useful.



Consider the diagram on the left:
Since: $A+B=x$, $A+D=w$, $B+C=y$ and $D+C=z$, we can see that $(A+B)+(D+C)=x+z$ as well as $(A+D)+(B+C)=w+y$.
A closer look reveals that: $x+z=w+y$.
Returning to q1, we now know that $24+x=18+25 \therefore x=19$. Looking at q2, we can now say that $25+x=21+23 \therefore x=29$.

Another interesting observation about this "general" case is this. $A+B+C+D+x+y+z+w$ (i.e. the total of all of the numbers being used in the puzzle) can be written as $(A+B+C+D)+(x+z)+(z+w)$ which is the same as $(x+z)+(x+z)+(x+z)$ or $3(x+z)$. This means that the overall total will always be a multiple of three! We will welcome this in a later puzzle.

Finally, we see that $x+y+z+w=2(A+B+C+D)$, which is similar to a result that we have seen before.

q3. BL

$i)\ 9+A=15 \therefore A=6$

$ii)\ A+B=12 \therefore 6+B=12 \therefore B=6$

$iii)\ B+8=y \therefore y=14$

$iv)\ 9+8=x \therefore x=17$

$Note: 12+x=15+y$

q4. BR

$i)\ 5+A=11 \therefore A=6$

$ii)\ A+8=y \therefore 6+8=y \therefore y=14$

$iii)\ 8+B=14 \therefore B=6$

$iv)\ 5+B=x \therefore 5+6=x \therefore x=11$

$Note: x+y=11+14$

These last two puzzles could be approached in a slightly different manner.

q3. BL

$x=9+8 \therefore x=17$

since we know that $12+17=15+y \therefore y=14$

$\therefore B=6$ and $A=6$.

q4. BR

$A=11-5 \therefore A=6$ and $B=14-8 \therefore B=6$

$\therefore y=6+8 \therefore y=14$

but $x+y=11+14 \therefore x+14=11+14 \therefore x=11$

I#5 **Sum:**

q1. TL

We have: $A+B=8$, $A+D=9$, $D+C=11$ and $B+C=10$. This is quite a different situation for us! Let us try to reduce the number of unknowns that we have to deal with. We will express each of the unknowns in terms of *B*. (We could just as easily used *A*, *C* or *D*.)

We get $A=8-B$, $C=10-B$ and $D=11-C$ which we change to $D=11-(10-B)$ or $D=1+B$. And, naturally, $B=B$. This yields the following system of equations.

$A=8-B$

$B=B$

$C=10-B$

$D=1+B$

By observing the rules for "cow-friendly" numbers, we can impose various restrictions on these unknowns: e.g.:

$A>0 \therefore 8-B>0 \therefore B\in\{1,2,3,4,5,6,7\}$

This will generate all possible solutions to the puzzle:

$$\begin{aligned}(A,B,C,D)&=(7,1,9,2)\\&=(6,2,8,3)\\&=(5,3,7,4)\\&=(4,4,6,5)\\&=(3,5,5,6)\\&=(2,6,4,7)\\&=(1,7,3,8)\end{aligned}$$

q2. TR

We know that $A+15=z$ and $B+15=w$, therefore, since $w=z$ $A+15=B+15$. This means that $A=B$.

Since $A+B=36$, $A=B=18$. Also $z=w=33$. $B+12=y \therefore y=30$. $12+A=x \therefore x=30$.

We should have been able to predict that *x* and *y* would be equal, since *w* and *z* were!

q3. BL

We know that $9+y=10+x \therefore y=1+x$ and that $x+y=7$. This means that $x=3$ and $y=4$.

If $x=3$, then *A* can be only 1 or 2 ("cow-friendly"!). Thus the only possible solutions to the puzzle are:

$$\begin{aligned}(A,B,C,D)&=(1,8,2,2)\\&=(2,7,3,1)\end{aligned}$$

q4. BR

Since $A+D=11$ and $D+C=17$, we see that $A+C+2D=28$ and since $A+C=12$, we have $12+2D=28$ or $D=8$. Thus: $A=3$, $C=9$, $B=12$ and $x=21$. We can verify this by checking that $11+x=15+17$.

I#6 **Sum:**

q1. TL

We see immediately that $x=25$ and $y=27$.

Since $x+y+z+w=2(A+B+C+D)$, we can see that $96=2(A+B+C+D) \therefore A+B+C+D=48$.

Substitution gives us $10+15+12+D=48 \therefore D=11$. This means that $w=21$ and $z=23$.

q2. TR

Since $x+y+z+w=2(A+B+C+D)$, we have $28+29+z+w=2(60) \therefore z+w=63$.

Also, since $28+z=29+w$, we see that $z=w+1$. This results in $z=32$ and $w=31$.
We notice that $A\in(1,2,3,...,27)$. Thus $B\in(27,26,25,...,1)$, $C\in(2,3,4,...,28)$ and $D\in(30,29,28,...,4)$.
To generate solutions to this puzzle, simple select the appropriate values from these sets. Three possibilities are $(A,B,C,D)=(1,27,2,30)$, $(A,B,C,D)=(2,26,3,29)$ and $(A,B,C,D)=(3,25,4,28)$.

q3. Bottom
In the bottom left puzzle we see that $35+36\neq 42+30$, thus there is no possible solution to this puzzle.
In the bottom right puzzle, we see that $37+33=40+30$. This means that it is possible to solve this puzzle.

There are many possible solutions: $A\in(36,35,...,8)$, $B\in(1,2,...,29)$, $C\in(29,28,...,1)$, $D\in(4,5,...,32)$.
Another way to show that the first puzzle is not possible is to consider the "parity" of *A*, *B*, *C* and *D*.
If $A+B=35$ then either *A* or *B* has to be an odd number. Assume for the moment that *A* is odd and *B* is even. Since $A+D=42$, it follows that *D* must be odd and since $D+C=36$, it follows that *C* must also be odd. Now, $B+C=30$, meaning that *B* also has to be odd. This leaves us with the result that B has to be both odd and even, which is obviously not possible.
If we had originally made the assumption that *A* was even, we would have ended up in a similar situation.

I#7 Sums

We see that $A=x-B$, $B=y-C$ and $C=z-A$. Combining these equations gives us $A=x-(y-(z-A))$.
This is the same as $2A=x-y+z$ or $A=\frac{x-y+z}{2}$. Parts *ii*) and *iii*) follow directly. You should pay particular attention to the symmetry in the puzzle as well as the algebra.

I#8 Consecutive Sums

Top Puzzle.
Consider that *x*, *y* and *z* are consecutive whole numbers. Let $x=x$, $y=x+1$ and $z=x+2$. Since we know that $x+C=y+A=z+B$ (see *I#2*), we can write $x+C=x+1+A=x+2+B$. Thus $B=A-1$ and $C=A+1$. This means that *A*, *B* and *C* are also consecutive whole numbers.

Similarly, consider that *A*, *B* and *C* are consecutive whole numbers. Let $A=A$, $B=A+1$ and $C=A+2$. Again we know that $x+C=y+A=z+B$, so we can write $x+A+2=y+A=z+A+1$. Thus we have $x=y-2$, $y=y$ and $z=y-1$. This means that *x*, *y* and *z* are also consecutive whole numbers. It's interesting to note that if the numbers *A*, *B* and *C* are ascending in clockwise order, *x*, *y* and *z* will be ascending in counter-clockwise order.

We say that *A*, *B* and *C* are consecutive whole numbers ***iff*** *x*, *y* and *z* are consecutive whole numbers. The notation ***iff*** is the abbreviation for *if and only if*. In mathematics, this notation is used to indicate the strongest sort of connection between two *conditional statements*. It says that $P\rightarrow Q$ (if *P* then *Q* or if *hypothesis* then *conclusion*) as well as $Q\rightarrow P$ (if *Q* then *P*). $P\rightarrow Q$ and $Q\rightarrow P$ are called *converse* statements because they reverse the *hypothesis* and the *conclusion*. $P\rightarrow Q$ is the *converse* of $Q\rightarrow P$. You will often see this written as $P\leftrightarrow Q$.

Bottom Puzzle.

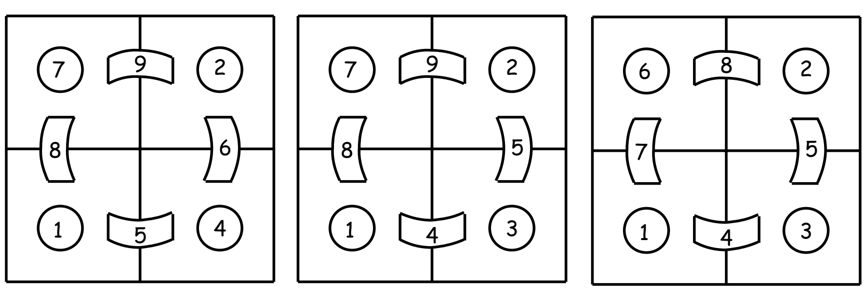
Consider that *A*, *B*, *C* and *D* are consecutive whole numbers. Let's replace these with, first of all *A*, $A+1$, $A+2$ and $A+3$ and secondly, with *A*, $A+1$, $A+3$ and $A+2$. Make sure that you understand that there are actually two (and only two!) cases for us to consider!
First case.
We can see that $x=2A+1$, $y=2A+3$, $z=2A+5$ and $w=2A+3$. These are obviously not consecutive whole numbers. In fact two of them are the same numbers!
Second case.
We can see that $x=2A+1$, $y=2A+5$, $z=2A+4$ and $w=2A+2$. These also are obviously not consecutive whole numbers.

How about the converse? Consider that *x*, *y*, *z* and *w* are consecutive whole numbers. Is it possible that *A*, *B*, *C* and *D* will also be consecutive whole numbers? Recall that $x+y=w+z$ must be true for this puzzle to work. But if *x*, *y*, *z* and *w* are consecutive whole numbers then this condition cannot possibly be met. Try using the numbers 1,2,3 and 4 to convince yourself of this.

I#9 Single Digits Sums

q1
Since we are using only eight different single-digit numbers for this puzzle, it is a good idea to examine which numbers can be left out of our considerations. We know that $A+B+C+D+x+y+z+w$ has to be a multiple of three (see *I#4*), while the available numbers (1,2,3,4,5,6,7,8,9) all sum to 45. This means that the number left out also has to be a multiple of three: ie. 3,6 or 9.

1. Consider the digits $\{1,2,4,5,6,7,8,9\}$ i.e. leaving out 3. These sum to 42.
If we recall that $A+B+C+D=x+z=w+y$, we can see that $w+y=x+z=A+B+C+D=14$ ($42\div 3$). Thus we can let $\{w,y\}=\{9,5\}$ and $\{x,z\}=\{8,6\}$. These are the only pairs that add to 14 and this leads us to the first solution.
2. Consider the digits $\{1,2,3,4,5,7,8,9\}$ i.e. leaving out 6. These sum to 39.
Thus $w+y=x+z=A+B+C+D=13$ and we can let $\{w,y\}=\{9,4\}$ and $\{x,z\}=\{8,5\}$.
This leads to the second solution.
3. Consider the digits $\{1,2,3,4,5,6,7,8\}$ i.e. leaving out 9. These sum to 36.
Thus $w+y=x+z=A+B+C+D=12$ and we can let $\{w,y\}=\{8,4\}$ and $\{x,z\}=\{7,5\}$.
This leads to the third solution.

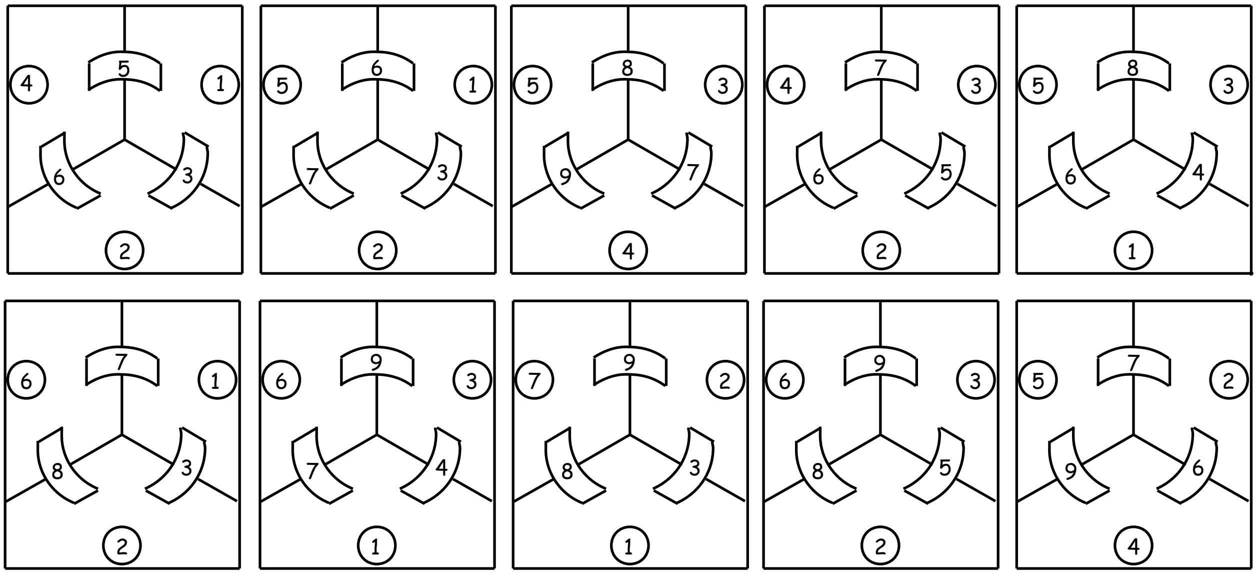


q.2
We know that $A+y=B+z=C+x$. This means that whichever six numbers we choose, they must come in 3 pairs that each have the same sum. This sum can only be 7,8,9,10,11,12 or 13 (why?).

Notice that $7=1+6=2+5=3+4$, $8=1+7=2+6=3+5$. Whereas $6=1+5=2+4$.

By experimenting, you will find that 13 doesn't work. 7, 8 and 12 yield one solution each. 9 yields 3 solutions, while 10 and 11 yield 2 solutions each. There are 10 different solutions in all.

Here they are:



This is a very difficult puzzle that requires a lot of careful analysis.

Of course, there are many other solutions but these will all be copies of one of the solutions shown above. It is interesting to think about just how many solutions there are, if we are to include all of the "copies". Consider one of the solutions shown above. In how many ways can you replicate this arrangement using rotations and refelctions?