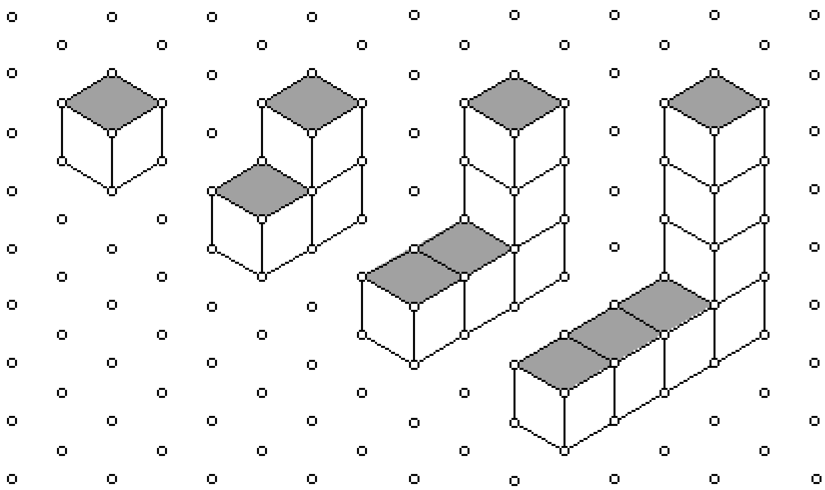
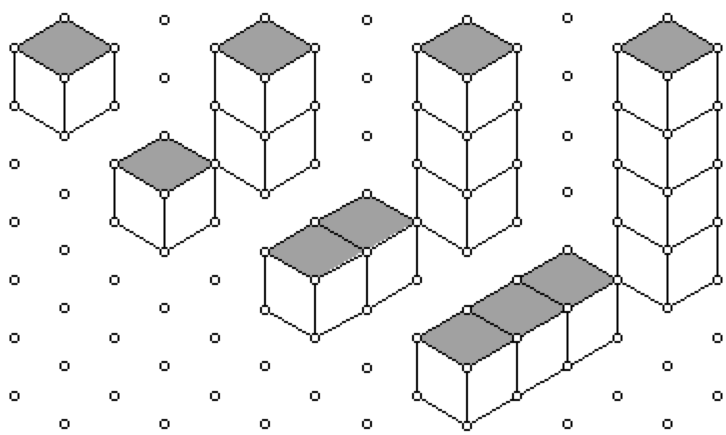
ALGEBRAIC MODELS

You may have noticed that there are several approaches that we could have taken in establishing the formulae for the total number of cubes in model sequences.

A: CONSIDER THE *GROWING L'S*



These models could be thought of as looking like this:



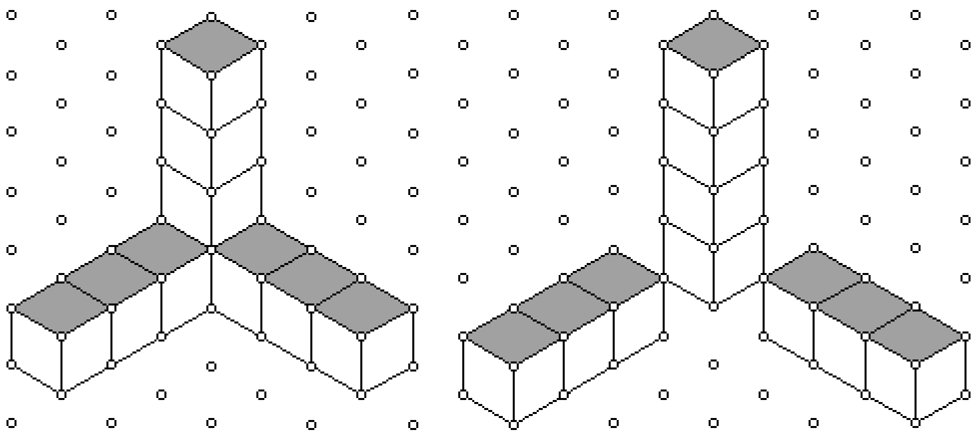
You can see that the models have been split creating two new sequences of models.

Model Number	1	2	3	4	…	N
Number of Cubes	1 + 0	2 + 1	3 + 2	4 + 3		$N+(N-1)$

Explain how the numbers in the second row of this table relate to the sequence of models shown above. **Compare the formula** for the Number of Cubes here with the formula that you found before. **Are they the same?**

B: <u>CONSIDER THE *SPROUTING ARMS*</u>

The fourth model in that sequence could be thought of as looking like this:



You can see that the model has been split creating three new models. The sequence would generate the following numbers.

Model Number	1	2	3	4	…	N
Number of Cubes	$1+2\times(0)$	$2+2\times(1)$	$3+2\times(2)$	$4+2\times(3)$		$N+2(N-1)$

Explain how the numbers in the second row of this table relate to the sequence of models shown above.

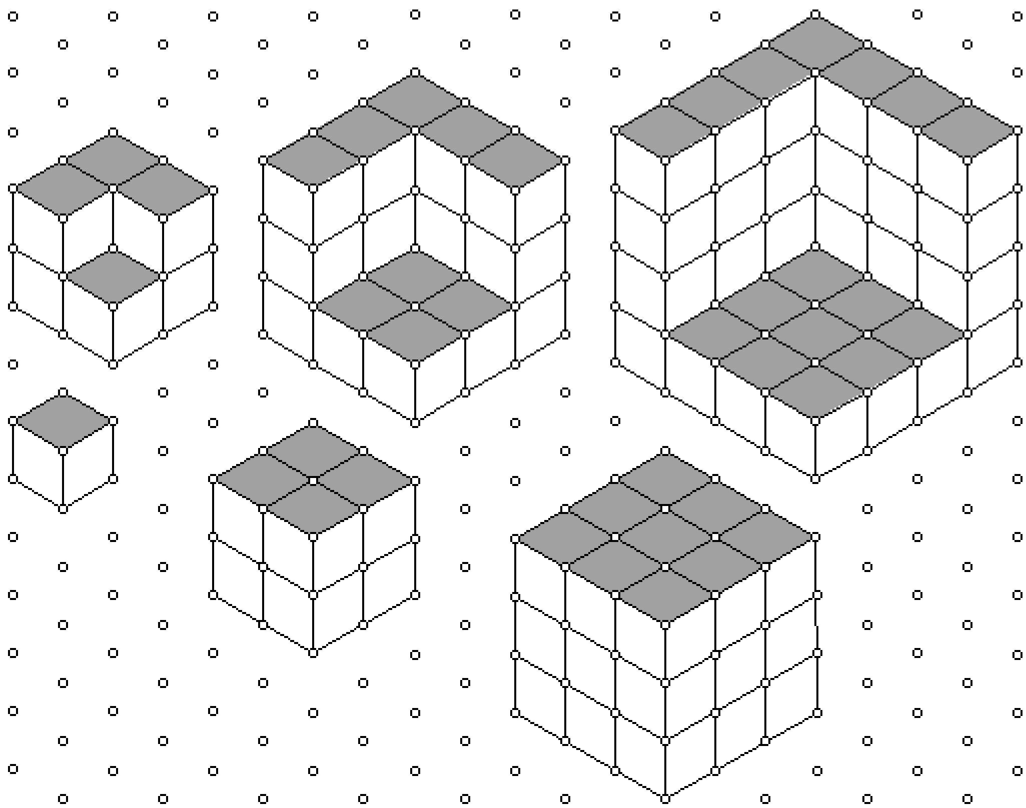
Compare the formula for the Number of Cubes here with the formula that you found before.

Are they the same?

Find alternate formulae for the numbers of cubes in the *Plus Pluses*, *Parading Pedestals* and *Picture This* sequences.

C: RE-CONSIDER THE *GROWING CORNERS*. *Part a)*

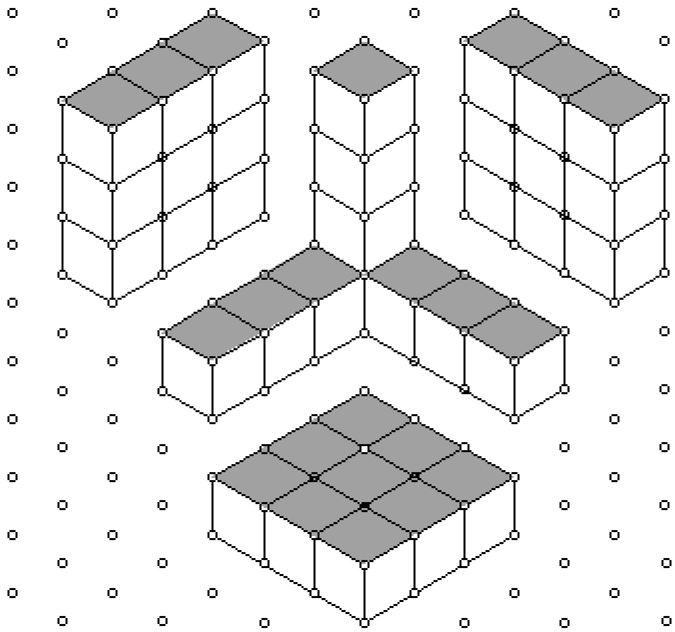
These models could be thought of as "cubes minus smaller cubes".



a) **Write a formula** that will give the number of cubes in the n^{th} model.
Your formula might look something like: $N = (\quad)^3 - (\quad)^3$.

C: RE-CONSIDER THE *GROWING CORNERS*: *Part b)*

The third model in this sequence can be thought of as looking like this:

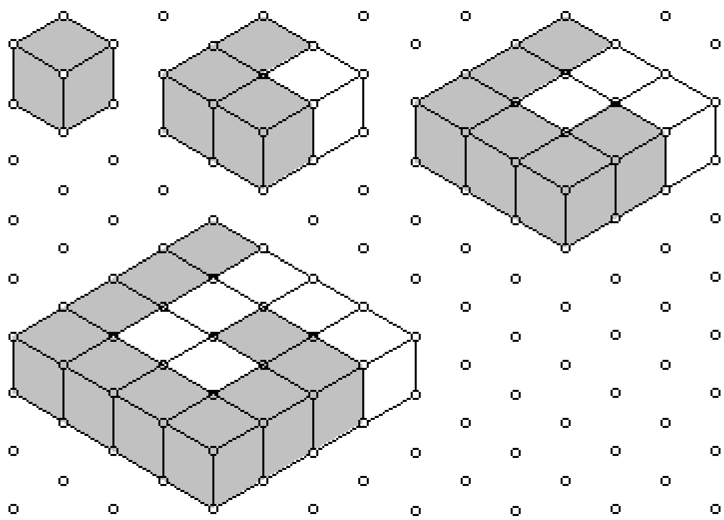


a) **Write a formula** that will give the number of cubes in each of the 4 parts in the n^{th} step in the sequence: *Growing Corners*. Your formula might look something like this: $N = 3(\quad)^2 + (\quad)$.

b) **Show that the formula** that you found in part a) of this question is actually the same as the formula that you found for part b) on page 39.

D: ***SNAKES***

a) You can form the following sequence of models by joining two separate sequences - a sequence of dark models and a sequence of light models. **Build the next model** in the sequence using cubes of two different colours. How many dark cubes will there be in the next model? How many cubes will there be in the next model?



b) The number of dark cubes in each model is given by the formula $N = \frac{n(n+1)}{2}$. **Explain why** this is the same as the formula found in the *Narrow Stairway* (page 35).

This formula can be expressed using function notation as: $f(n) = \frac{n(n+1)}{2}$.

c) You will notice that the number of light cubes in each model is the same the number of dark cubes in the *previous model*. Using function notation, we can say that the number of light cubes in each model is given by the function $f(n) = \frac{(n-1)n}{2}$.

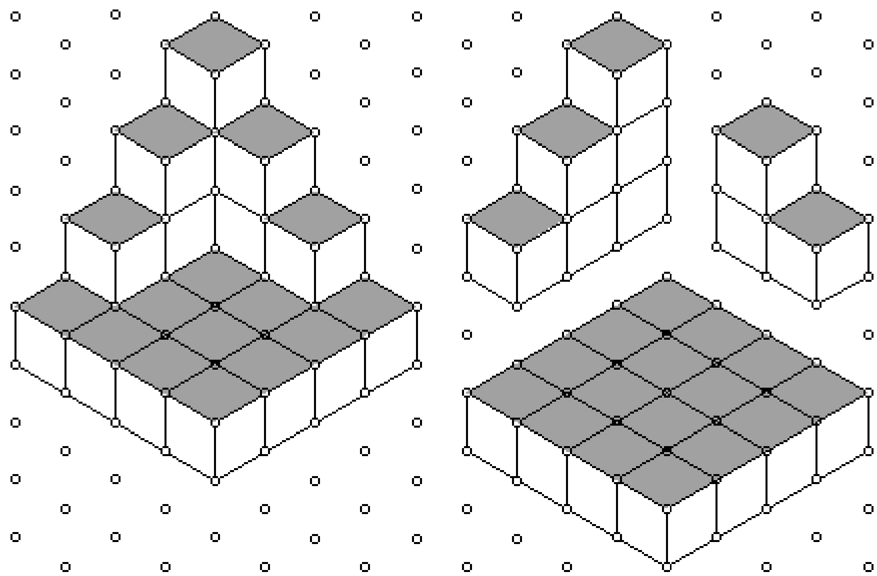
Explain why this function works.

d) **Explain why the following equation is always true**:

$$n^2 = \frac{n(n+1)}{2} + \frac{(n-1)n}{2}$$

E: ***STEPPED CORNER***

The following model is the fourth in a sequence of models.



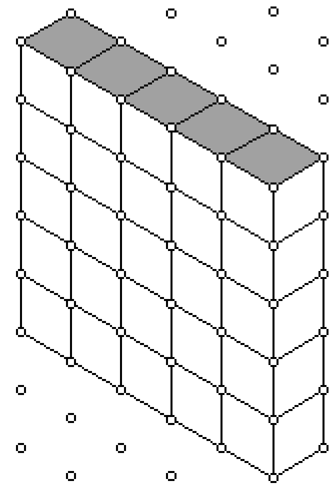
a) **Build the first four models** in this sequence using cubes. **Describe the relationships** between successive models.

b) **Build a function** that expresses the number of cubes, *N*, in each model in terms of *n*, the model number.

c) **Re-write the function that you found in part b)**, as the sum of two squares. Describe how this sum can be "seen" in the above model.

F: ***THE BIG SQUARE***

This model can be viewed as the fifth in a sequence. For the following questions think of it as the n^{th} model in a sequence.



Imagine that the entire model has been dipped into a can of green paint and after the paint has dried, the model has been broken into n^2 individual cubes. Some of these cubes will have paint on two faces; some will have paint on three faces and some will have paint on four faces.

a) How many cubes in the fifth model would have paint on two faces?
b) How many cubes in the fifth model would have paint on three faces?
c) How many cubes in the fifth model would have paint on four faces?

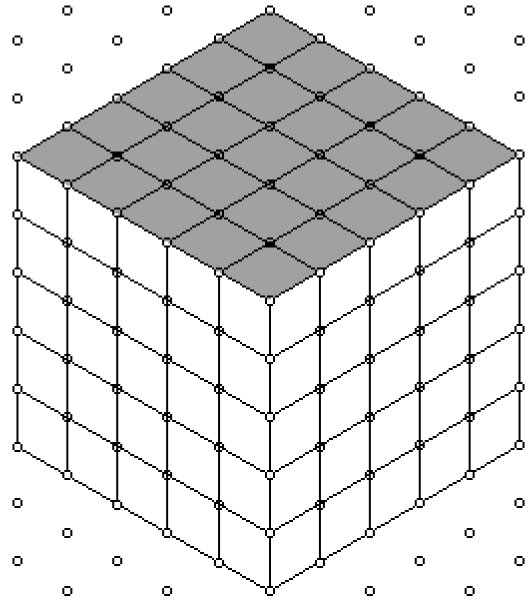
d) How many cubes in the n^{th} model would have paint on two faces?
e) How many cubes in the n^{th} model would have paint on three faces?
f) How many cubes in the n^{th} model would have paint on four faces?

g) **Explain why** the following equation must always be true:

$$n^2 = (n-2)^2 + 4(n-2) + 4$$

h) With a small square model, the number of cubes with paint on three faces would make up a relatively large part of the overall number, while the number of cubes with paint on two faces would be relatively small.
   i) **Create a function** that expresses the number of cubes with two painted faces as a percent of the total number of cubes. Graph this function for $2 \le n \le 10$.
      Why do we start with n = 2?
   ii) **Describe the graph and explain its shape**.

G: ***THE BIG CUBE***



This model can be viewed as the fifth in a sequence.

Imagine that the outside of this model has been painted green. Then the model has been broken into 125 individual cubes. Some of these cubes will have paint on one face; some will have paint on two faces; some will have paint on three faces, and some will no paint on any of their faces.

a) **How many** of the 125 cubes will have paint on just the one face? Two faces?

b) **How many** of the 125 cubes will have paint on three faces? Zero faces?

For the following questions you can think of this model as the n^{th} model in a sequence.

c) **Explain why** the following equation is true for all values of $n \geq 2$.
$n^3 = (n-2)^3 + 6(n-2)^2 + 12(n-2) + 8$. **Explain your thinking**.

d) **Describe the shape** modelled by the expression $n^3 - (n-2)^3$. Explain your thinking.

Think…

e) Is the following equation true? Explain your thinking. $6n^2 = n^3 - (n-2)^3$

f) If the previous equation is not true, how might you correct it? Explain your thinking.

g) With a small model, the number of cubes with paint on no faces would make up a small part of the overall number, while the number of cubes with paint on two faces would be larger.

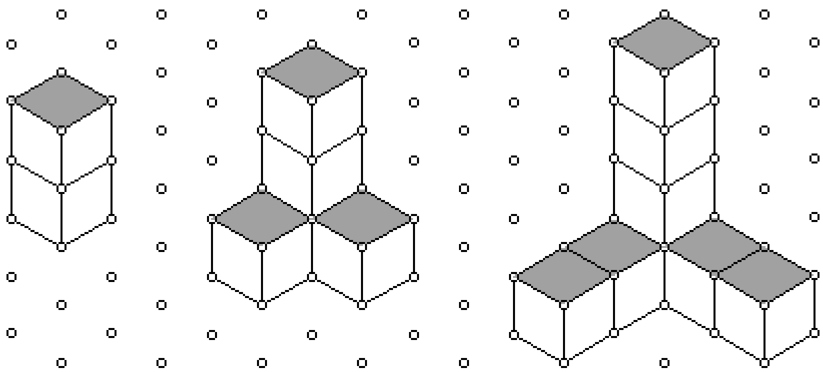
- **i) Create a function that expresses the number of cubes with zero painted faces as a fraction of the total number of cubes. Graph this function for** $2 \leq n \leq 25$**.**
- **ii) Create a function that expresses the number of cubes with one painted face as a fraction of the total number of cubes. Graph this function for** $2 \leq n \leq 25$**.**
- **iii) Create a function that expresses the number of cubes with two painted faces as a fraction of the total number of cubes. Graph this function for** $2 \leq n \leq 25$**.**

Why do we start with *n* = 2? Describe these graphs and explain their shapes.

H: ***STROLLING BONES AGAIN***)

a) **Build each model** in the following sequence.

b) **Determine a formula** for the total number of cubes in each model.



c) **Build 4 copies** of each model in this sequence and assemble these copies to create a sequence of skeleton cubes.

d) **Compare the formula** that you found in question b) to the formula that you previously found for the total number of cubes in each model of the *Strolling Bones* sequence. **How are they related? How are they different?**

e)

Think:

i) Determine a formula for the number of faces on each of the models shown in question b).

ii) Compare the formula that you found in part i) to the formula that you previously found for the total number of faces in each model of the *Strolling Bones* sequence (page 26).

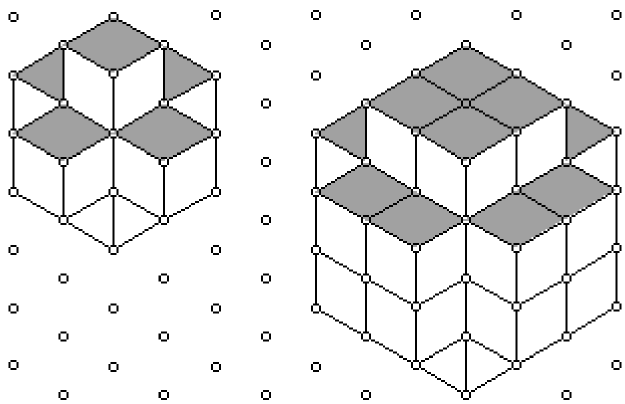
iii) You might think that by multiplying the formula that you found in part i) by 4, you would get the formula for the number of faces

I: ***INNER BONES: THE INSIDES OF THE STROLLING BONES***.

a) **Describe the next model** in the following sequence.

b) **Predict the number** of cubes that will be in the next two models.

c) **Build a formula** that will give the number of cubes in the n^{th} model.



Think…

Build and sketch a sequence of models using cubes in which the number of cubes in each model is given by the formula:

a) $N = 2n^2 + 4n$

b) $N = 2(n+1)^2 + (n+1)(n-1)$

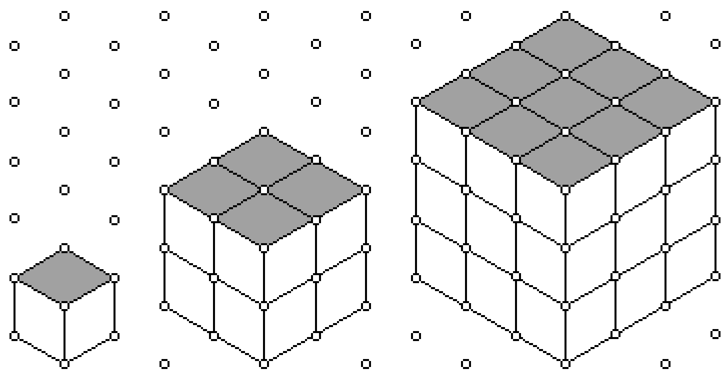
c) $N = (n+1)^2 + 4n$

d) $N = n^3 + n^2$ **or** $N = n^2(n+1)$

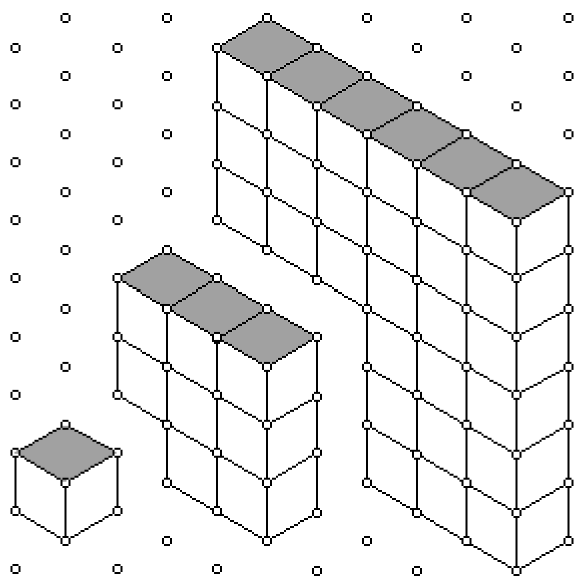
e) $N = n^3 + (n+2)^2$

J: ***SUM OF CUBES***

a) **Build the following sequence** of models using cubes.



b) Reassemble the models to form the following sequence of models.



c) Does this pattern hold for the next model in the first sequence?

d) **Explain why** the following equation is true for any positive integer *n*.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(1 + 2 + 3 + \ldots + n\right)^2$$