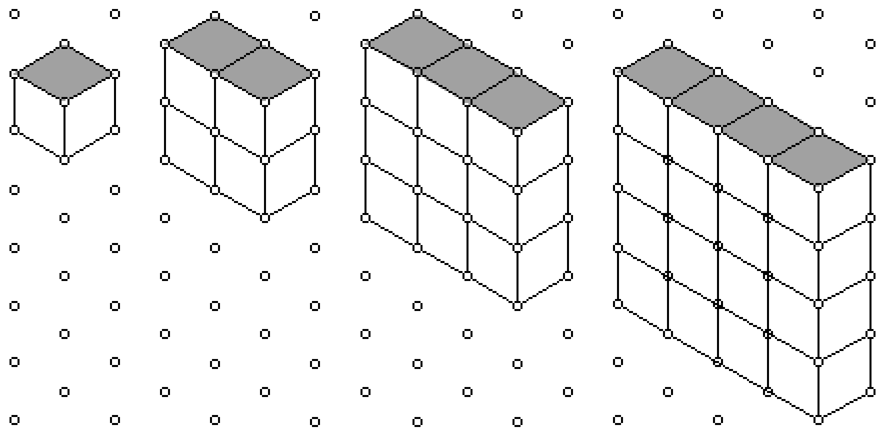
QUADRATIC MODELS – PATTERNS AND ALGEBRA

A: GROWING SQUARES

Consider the following sequence of models:



a) How many cubes will there be in the next model?

b) **Describe** how you would create the fifth model in this sequence by adding cubes to the fourth model.

c) How many cubes would you add to the fourth model to create the fifth model?

d) **Describe a pattern** in the numbers of cubes that are added at each step to form the next model. Refer to the "*Growing L's*"

e) How many cubes would you add to the eighth model to create the ninth model? Explain your thinking. **Explain how** you might use the function determined for the number of cubes in the "*Growing L's*" sequence.

f) Complete the following table:

model number *n*	1	2	3	4	5	6
number of cubes *N*	1	4				
first differences						
second differences						

In the preceding table you have been asked to complete two rows of differences. For some number patterns, the first row of differences will be constant. For other number patterns, you may have to construct more and more rows in order to find a row of constant differences. Some number patterns will never produce a row of constant differences. As with most of the number patterns in this chapter, this particular sequence will produce second differences that are constant. This will indicate that ***N*** is a quadratic function of ***n***.

In general, a quadratic function has the form $N = an^2 + bn + c$. The following are examples of quadratic functions: $N = 3n^2 + 1$, $N = n^2 + 3n - 4$ and $N = n^2 + 5n$.

1. a) **Compete the following table** as if ***N*** were a linear function of ***n***:

model number *n*	1	2	3	4	5	6
number of cubes *N*	3	4				
first differences	1					
second differences						

2. b) **Compete the following table** as if ***N*** were a quadratic function of ***n***:

model number *n*	1	2	3	4	5	6
number of cubes *N*	3	4				
first differences	1					
second differences		1				

3. What is the least number of terms in a sequence that you must know to be able to decide if it is generated by a linear function or by a quadratic function?

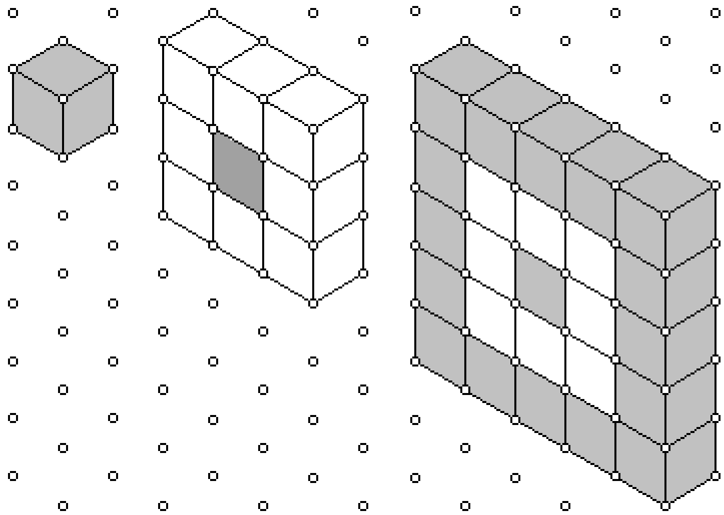
4.

> ***Think:***
> **Examine the following table and discuss any possible functions that might relate *N* to *n*.**

model number ***n***	1	2	3	4	5
number of cubes ***N***	1	2	4	8	…
first differences	1	2	4	…	
second differences		1	…	…	…
…		…	…	…	

B: ***SKIPPING SQUARES***

Consider the following sequence of models. **Build these models** using two different coloured cubes.



a) **Describe** how you would build the fourth model based on the third model.

b) How many cubes would you add to the third model in order to create the fourth model?

c) **Describe a pattern** in the numbers of cubes that are added at each step.

d) How many cubes would you add to the eighth model in order to create the ninth model? **Explain your thinking**.

e) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*	1	9				
first differences						
second differences						

Your completed table should look like this:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*	1	9	25	49	81	121
first differences	8	16	24	32	40	
second differences		8	8	8	8	8

The second differences in this table are all the same. This would indicate that the number of cubes, ***N***, is a quadratic function of the model number, ***n***. This function has the form $N = an^2 + bn + c$.

f) Find the function that expresses ***N*** in terms of ***n***.

Solution a)

From the table you can see that when $n=1, N=1$. When $n=2, N=9$. When $n=3, N=25$. If you substitute these values into the function $N = an^2 + bn + c$, you will get the following system of equations:

$$1 = a + b + c \quad (1)$$
$$9 = 4a + 2b + c \quad (2)$$
$$25 = 9a + 3b + c \quad (3)$$

$(2)-(1)$ yields $\quad 8 = 3a + b \quad (4)$

$(3)-(1)$ yields $\quad 24 = 8a + 2b$ or $12 = 4a + b \quad (5)$

$(5)-(4)$ yields $\quad 4 = a$

(4) yields $\quad 8 = 3(4) + b$ or $-4 = b$

(1) yields $\quad 1 = (4) + (-4) + c$ or $1 = c$.

Therefore the function is $N = 4n^2 - 4n + 1$.

You can verify that this function is correct by substituting $n = 4$ to see if $N = 49$.

Solution b)

If you look closely at the table, you might recognize the following pattern:

Model Number ***n***	1	2	3	4	5	6
Number of cubes ***N***	$1=1^2$ $=(2\times1-1)^2$	$9=3^2$ $=(2\times2-1)^2$	$25=5^2$ $=(2\times3-1)^2$	$49=7^2$ $=(2\times4-1)^2$	$81=9^2$ $=(2\times5-1)^2$	$121=11^2$ $=(2\times6-1)^2$

This pattern would indicate that the function that expresses ***N*** in terms of ***n*** is $N = (2n-1)^2$. If you expand this new expression, you will find that $N = 4n^2 - 4n + 1$ as before.

You should always be on the lookout for patterns!

g)

Think…

You have seen that $N = 4n^2 - 4n + 1$. The "next" model in the sequence will contain $4(n+1)^2 - 4(n+1) + 1$ cubes. Explain this last expression and determine a formula that generates the first differences.

Think… h)

Complete the following table where the generating formula for *N* is the general function $N = an^2 + bn + c$

Model Number *n*	1	2	3	4
Number of cubes *N*	$a+b+c$	$4a+2b+c$		
first differences				
second differences				

Think… i)

The general quadratic function is $N = an^2 + bn + c$. Using the following expressions, show that the second differences will always be constant.

$(a)\quad a(p-1)^2 + b(p-1) + c$

$(b)\quad ap^2 + bp + c$

$(c)\quad a(p+1)^2 + b(p+1) + c$

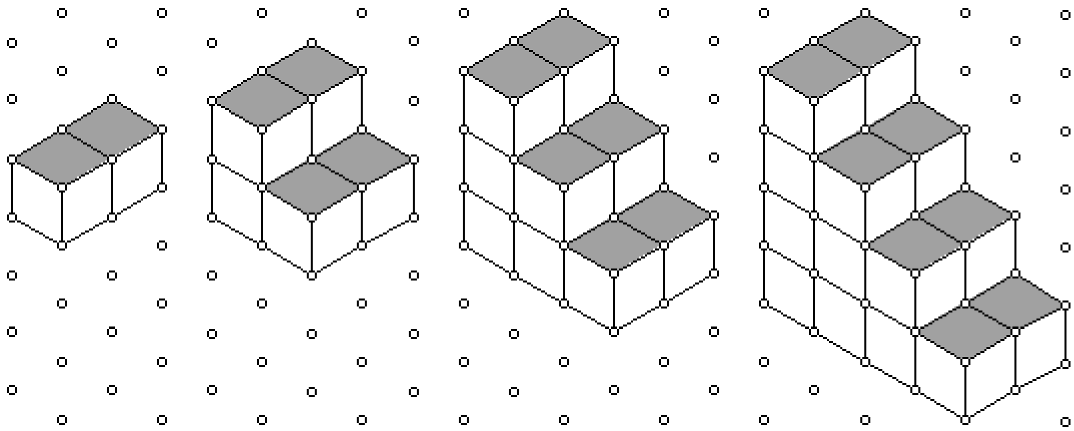
a) Explain how these expressions represent three consecutive numbers in a sequence that has been generated by a quadratic function.

b) Determine the second differences.

c) Describe the significance of your findings.

C: ***RISING STAIRWAY***

Consider the following sequence of models:



a) **Describe** how you would build the fifth model based on the fourth model.

b) How many cubes would you add to the fourth model in order to create the fifth model?

c) **Describe a pattern** in the numbers of cubes that are added at each step.

d) How many cubes would you add to the eighth model in order to create the ninth model?

e) What sort of function will relate ***N*** to ***n*** in this example? **Explain your thinking**.

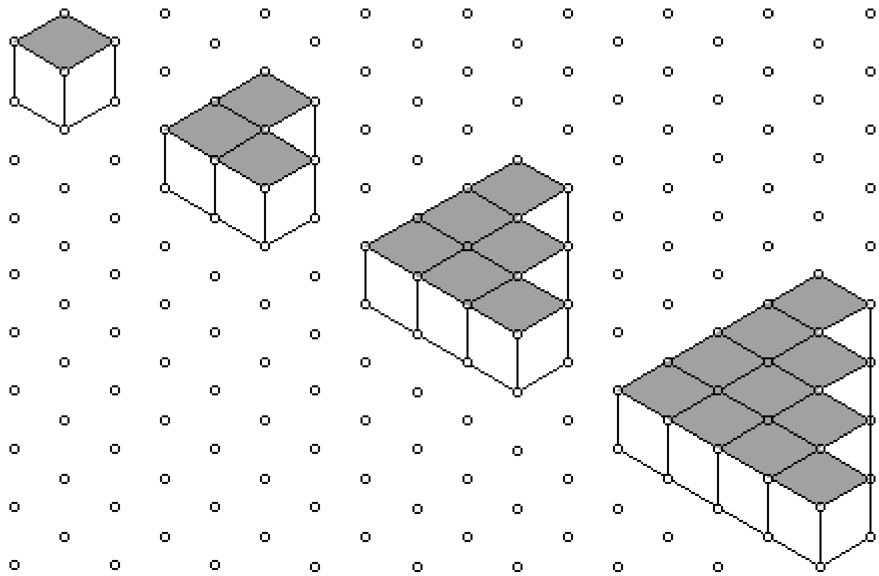
f) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*	2	6				
first differences						
second differences						

g) **Determine the function** that expresses *N* in terms of *n*.
   i) Use the method of ***Solution a)*** in section *B:* ***Skipping Squares*** .
   ii) Look for a pattern in the values of *N*. Try factoring these values.

D: ***NARROW STAIRWAY***

Consider the following sequence of models:



a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*	1	3				
first differences						
second differences						

b) **Determine the function** that expresses N in terms of n.
   i) Use the method of ***Solution a)*** in section *B: Skipping Squares*.
   ii) **Transform the function** found in the previous exercise.

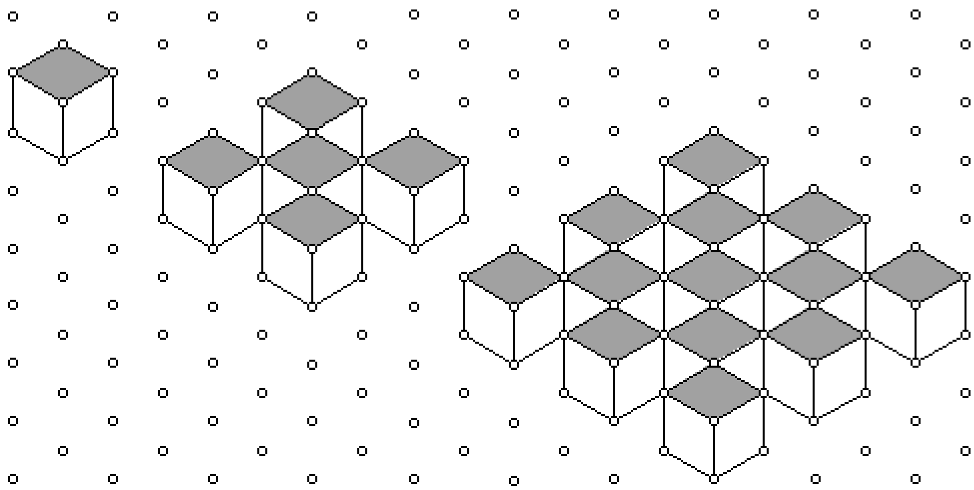
c) Use the function that you have determined in this example to find the following sums.
   Explain your thinking.

   i) $1+2+3+4+...+25$ ii) $1+2+3+4+...+100$

   iii) $2+4+6+8+...+100$ iv) $40+41+42+43+...+85$

E: ***CHECKER BOARD***

Consider the following sequence of patterns:



a) **Describe** how you would create the next two patterns in this sequence.

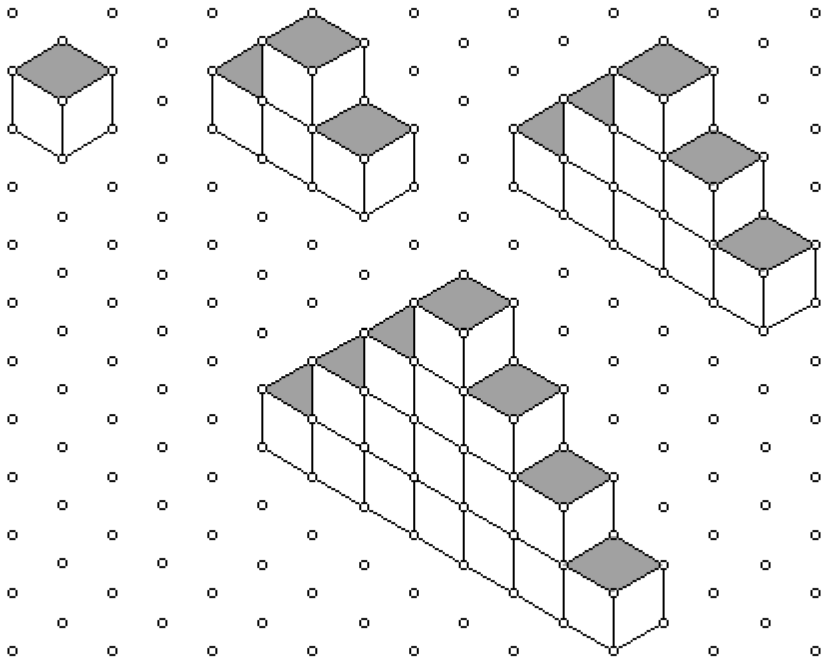
b) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						

c) **Determine the function** that expresses N in terms of n.
   i) Use the method of ***Solution a)*** in section *B: Skipping Squares*.
   ii) **Look for a pattern** in the values of N. You might want to consider the sum of two consecutive squares.

F: ***TROOPING TRIANGLES***

Consider the following sequence of models. **Build these models** using cubes.



a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						

b) **Determine the function** that expresses N in terms of n.

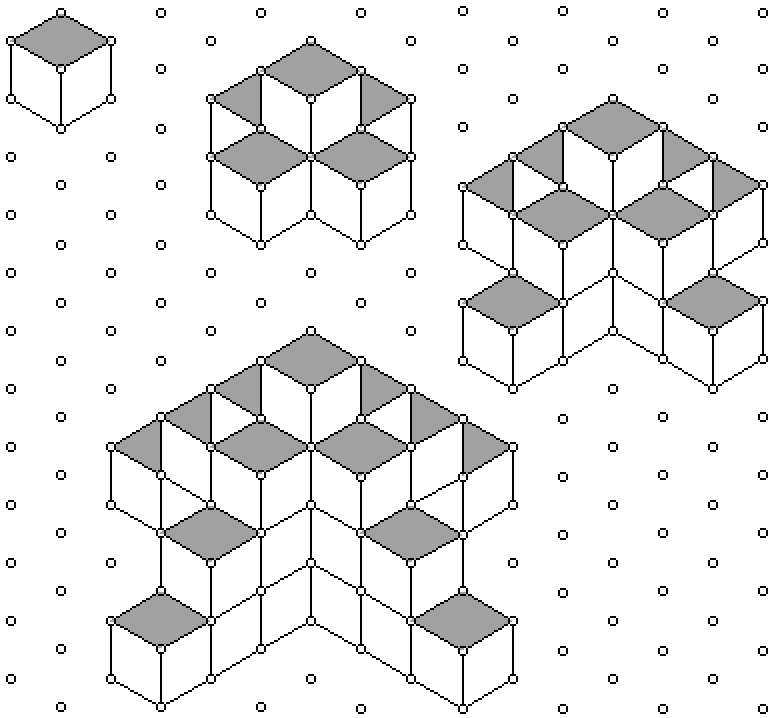
i) Use the method of *Solution a)* in section *B: Skipping Squares*.

ii) **Look for a pattern** in the values of N. **Try reassembling your models** to create new shapes that are easier to handle algebraically.

c) "The sum of the first k odd numbers is a perfect square". **Explain what this statement means. Explain why it is true**. What can you say about the sum of the first k even numbers? **Explain your thinking.**

G: ***TROMPING TOWERS***

Consider the following sequence of models:



a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						

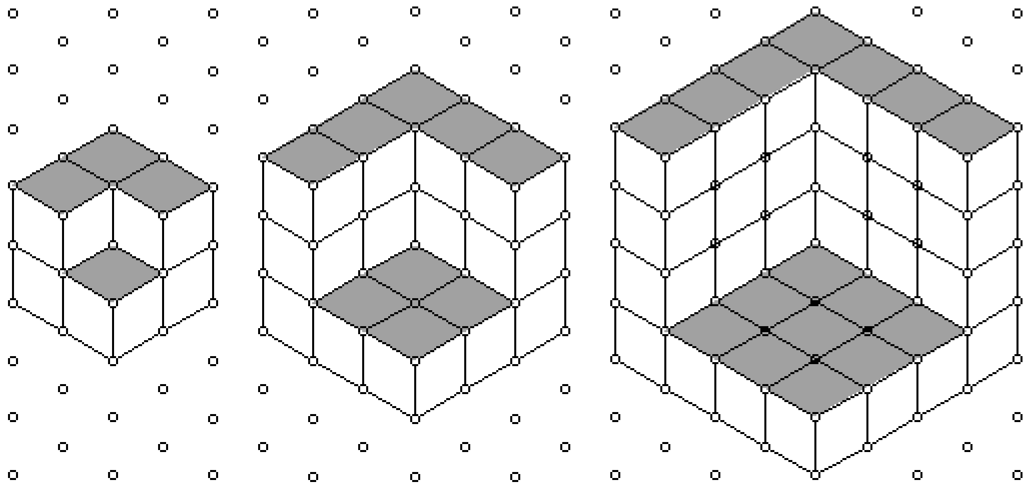
b) Determine the function that expresses *N* in terms of *n*.

i) Use the method of *Solution a)* in section *B: Skipping Squares*

ii) **Describe how these models are related to the *Trooping Triangles*. Describe how the function that you determined in that example can be transformed into a function that will describe the *Trooping Towers.***

H: ***GROWING CORNERS***

Consider the following sequence of models:



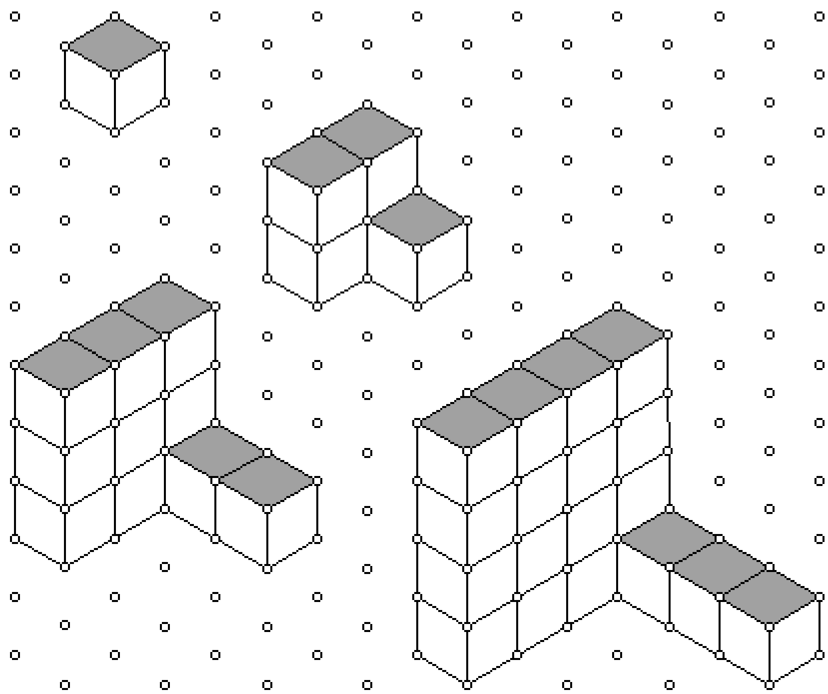
a) **Complete the following table**:

Model Number ***n***	1	2	3	4	5	6
Number of cubes ***N***						
first differences						
second differences						

b) Determine the function that expresses *N* in terms of *n*.
   i) Use the method of *Solution a)* in section *B: Skipping Squares*.
   ii) **Look for a pattern** in the values of *N*. **Describe the connection** between the sequence *Sprouting Arms* and this sequence. **Explain how** the function that you determined for the *Sprouting Arms* can be modified to describe the *Growing Corners*.

I: ***WANDERING WALLS***

Consider the following sequence of models:



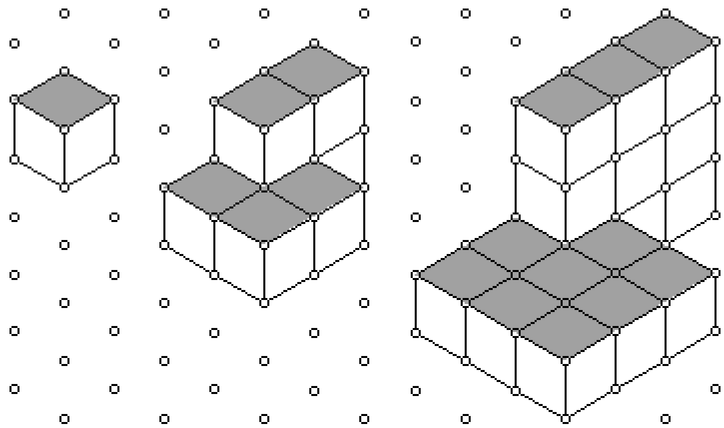
a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						

b) Determine the function that expresses N in terms of n.
   i) Use the method of *Solution a)* in section *B: Skipping Squares*.
   ii) **Look for patterns** in the values of N that appear to correspond to the shape of the models.

J: ***WALLS 'N' FLOORS***

Consider the following sequence of models:



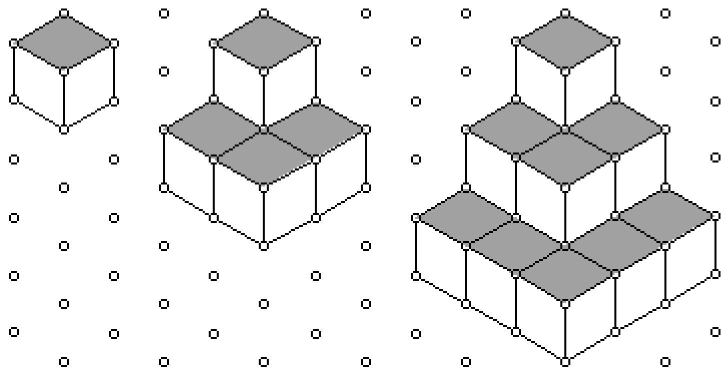
a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						

b) Determine the function that expresses N in terms of n.
   i) Use the method of *Solution a)* in section *B: Skipping Squares*.
   ii) **Transform the function found in the previous exercise**.

K: ***PARADING PYRAMIDS…CHALLENGE!***

Consider the following sequence of models:



a) **Describe how** you would add cubes to the third model to create a fourth model.

b) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						
third differences						

c) To obtain each new model, a "square" has to be added. This means that the first differences are generated by a quadratic function. **Explain what this means**.

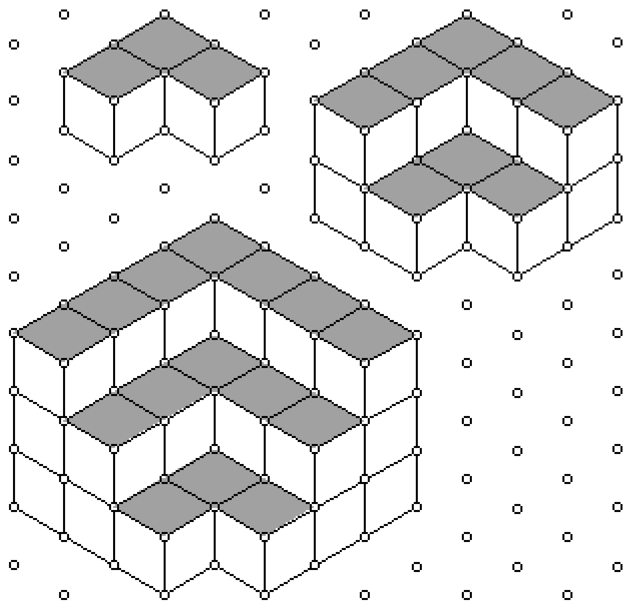
d) What sort of function will generate the second differences? **Explain your thinking.**

e) What sort of function will generate the third differences? **Explain your thinking.**

f) Determine the function that expresses N in terms of n.
Use the method of *Solution a)* in section *B: Skipping Squares*.

g)

> ***Think:***
> **If the *Parading Pyramids* were combined with the following models, a sequence of solid blocks would be formed. Determine the function that will generate the number of cubes in each of these blocks and using subtraction, determine the function that expresses *N* in terms of *n* for the following sequence of models.**

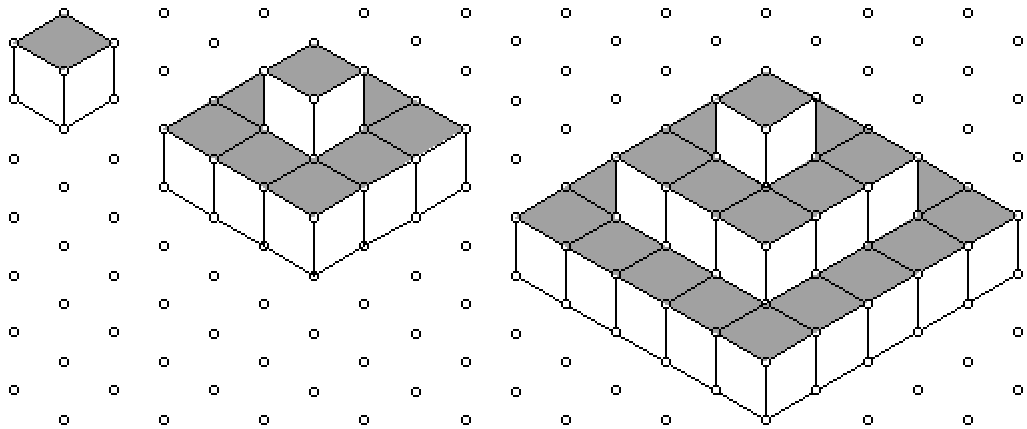


h)

> ***Think:***
> **Verify your answer in part g) by determining this function directly.**

L: ***WALKING WEDDING CAKES…CHALLENGE!***

Consider the following sequence of models:



a) **Complete the following table**:

Model Number *n*	1	2	3	4	5	6
Number of cubes *N*						
first differences						
second differences						
third differences						

b) **Determine the function** that expresses N in terms of n.

Projects:

1. In the sequence *Plus Pluses*, some of the models can be reassembled to form squares. Which ones? Describe any patterns that there may be in the numbers of the models that can be rearranged in this fashion. Explain fully and convincingly how you can be sure of your answer.

2. In the sequence *Sprouting Arms*, some of the models can be reassembled to form squares. Which ones? Describe any patterns that there may be in the numbers of the models that can be rearranged in this fashion. Explain fully and convincingly how you can be sure of your answer.

3. In the sequence *Parading Pedestals*, some of the models can be reassembled to form squares. Which ones? Describe any patterns that there may be in the numbers of the models that can be rearranged in this fashion. Explain fully and convincingly how you can be sure of your answer.

4. For what values of q will the expression $8q+1$ generate perfect squares?

5. Using cubes, build a sequence of four models that can be represented by each of these expressions.

a) $3q+1$ b) $3q-2$ c) c) $3q+4$

When reassembled, these models will sometimes form perfect squares. For which values of *q*, is this possible? Describe any patterns that you find. Explain fully and convincingly how you can be sure of your answer.

6. The above questions could easily be modified to read "perfect cubes" rather than "perfect squares". Choose a question and explore it for "perfect cubes".