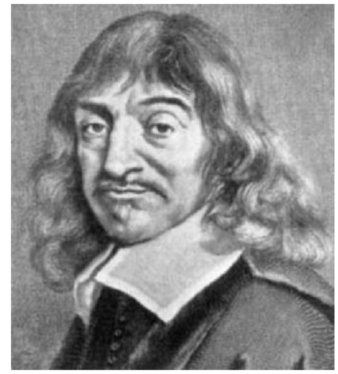


In 1637 René Descartes published a book that was to become one of the most important texts in all of mathematics.

In that book, *Des matieres de la Geometrie*, Descartes showed how to create geometric models of a variety of arithmetic and algebraic operations. He discussed some algebraic solutions to a variety of geometric problems, as well as some geometric solutions to a number of algebraic problems. By combining these great threads of mathematical thinking, he created the ground work for pretty well all of modern day mathematics. When you remember that historically, geometry was given philosophical primacy over other forms of mathematical reasoning, this bold new approach was quite revolutionary.

This article offers a brief view of that book through the eyes of a modern day enthusiast of the Geometer's Sketchpad. Most of the figures included in this article are available as GSP files.

We'll start by considering some "dynamic" arithmetic. It might be helpful to reflect on the idea of a square root. No doubt, the square root of *N* is understood to be the number that results in *N* when it is multiplied by itself. Thus, even if we have no problem saying that the square root of 16 is 4, what specific value can we attach to the square root of 7? A calculator or an algorithm can give an approximation: however, there is no algorithm that will give us the exact square root of 7. However, it is possible to create a geometric construction of the square root of 7 - not an approximation but the actual thing!

Book 1, of Descartes' *Geometrie*, is titled: *Problems the Construction of which requires only Lines and Circles.* The headings of the first two sections are: *How the calculations of arithmetic are related to the operations of geometry* and *How multiplication, division, and the extraction of square roots are performed geometrically*.

Let's examine some of these "calculations" starting with the simplest: addition. Can we model addition using the Geometer's Sketchpad? Not surprisingly, we can.

We need to understand that in arithmetic the objects are numbers, in geometry we are going to use line segments. We need to find a bridge between numbers and line segments. This bridge will be *length*.

First, create the two variable line segments to be added: **a** and **b**.

Next, create a ray **P** through **Q**, upon which we will create a line segment whose length is equal to the sum of the lengths of segments **a** and **b**. Create a circle with centre at **P** and radius **a**. Label the point where this circle intersects the ray **PQ**, **M**. Create a second circle with centre **M** with radius **b**. Label the point where this circle intersects the ray **PQ**, **N**. The segment **PN** will have the same length as the sum of lengths **a** and **b**.

If you alter the lengths of **a** or **b**, you will see a corresponding change in the length of **PN**.

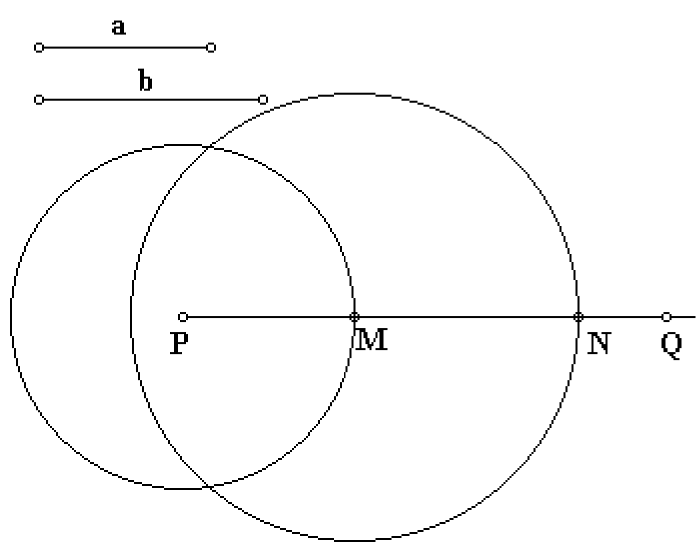


Fig. 1

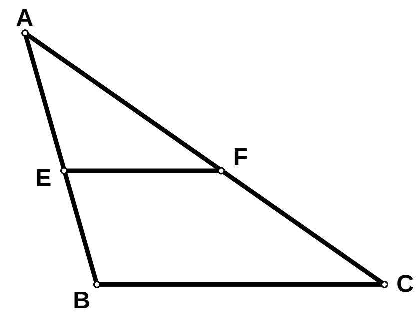
Constructing a geometric model for subtraction is quite straightforward.

After addition and subtraction, we can consider multiplication and division. Given two line segments, we want to construct a third segment with a length equal to the product of the first two. This requires some facility with the geometry of a triangle and a bit of creative thinking.

Multiplication and proportionality should always be thought of together. For our problem, we need to create a proportionality involving two required lengths and their product.

Consider the triangle *ABC*, with *E* and *F* on sides *AB* and *AC* such that $EF \parallel BC$.

By considering similarities, we can show that $AE:EB = AF:FC$. Thus: $AE \times FC = EB \times AF$. If we create this diagram so that the length of AE is 1 unit, we see that $FC = EB \times AF$. It is quite easy to create a geometric model to reflect this.



Ensure that *AE* is one unit in length. In the triangle ABC, the segments *AE*, *EB* and *AF* were created using circles on hidden rays. *BC* was created parallel to *EF*. The accuracy of this model should not be judged by the measurements displayed by the GSP (which aren't all that bad considering that we can see only four digits for *EB* and *AF*!).
We *know* that $FC = EB \times AF$.

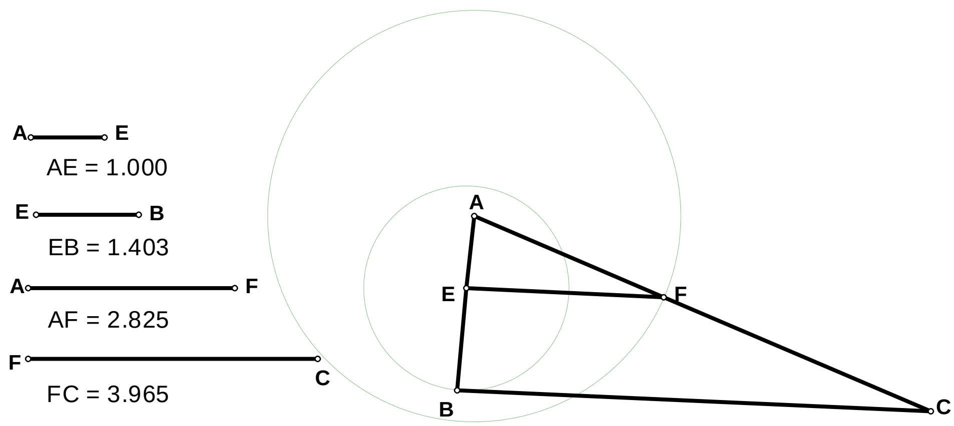


Fig. 2

See if you can create the following geometric model for the *division* of two numbers.

Note: $EB = \dfrac{FC}{AF}$.

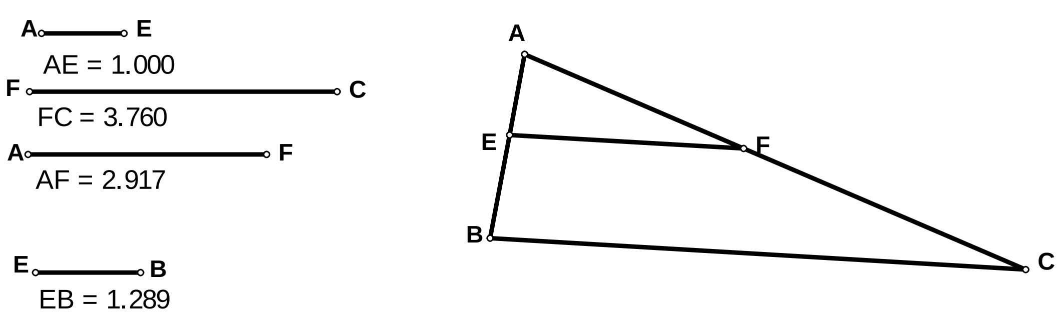
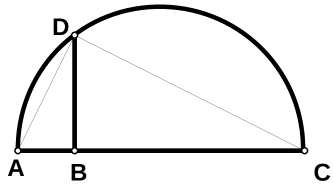


Fig. 2

The next construction offered by Descartes models the extraction of the square root of a number.

In this geometric model, *AB* has been constructed with unit length. *BC* is of variable length with *C* constructed on the (hidden) ray *A* through *B*. *AC* is a diameter of the circle *ADC*. *BD* is perpendicular to *AC*. *BD* is the mean proportional between *AB* and *BC*. (The diagram on the right might help to clarify this. Recall: $\angle ADC = 90°$ and $\Delta ADB \sim \Delta DCB \sim \Delta ACD$)



We have the following:

$$\frac{AB}{BD} = \frac{BD}{BC}$$
$$\therefore AB \times BC = BD^2$$
$$\therefore BD = \sqrt{BC}$$

Thus, *BD* is the square root of *BC*.

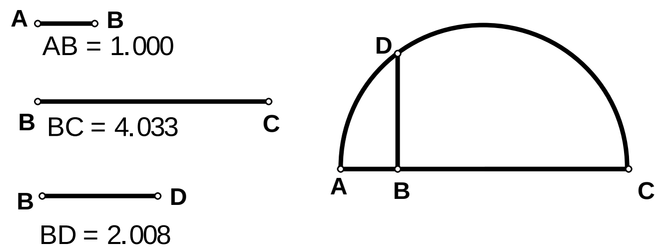


Fig. 5

This model can be modified to generate a parabola.

DP and *CP* have been constructed perpendicular to *BD* and *BC* respectively. Since $DB^2 = AB \times BC$ $AB = 1$ and $BC = DP$, we have $DB^2 = DP$. The parabola is the locus of *P* with *C* as the handle.

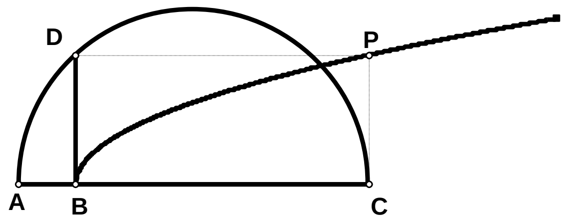


Fig. 6

If we allow *AB* (a) to vary and assign the length of *DB* to y and the length of *BC* to x, then the parabola will have the equation $y^2 = ax$.

This construction is entirely different from the ones we find using more "traditional" Euclidean means (i.e. with focus and directrix). It should be noted that this approach is very close to what we think of as "analytic geometry". However, the resulting equation does not make any reference to an axis system. The equation simply relates various quantities within the geometric construction. In this sense it is much closer to the Euclidean tradition than what we might now call "Cartesian".

Here's a nice consequence of this construction. We can now construct cube roots!

If we create two perpendicular parabolas as shown, u, the first coordinate of Q, the point of intersection, is the *cube root* of the length a. After you create the geometric model, you should have a go at proving this result. (see note 1)

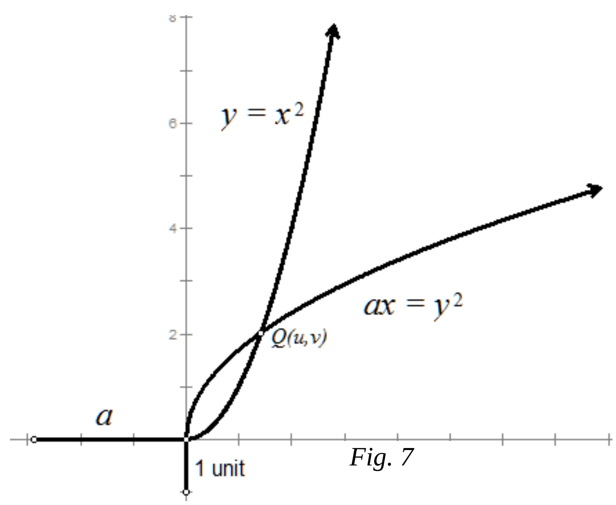


Fig. 7

Descartes was not the first person to write about this model. Omar Khayyam described a similar method in about 1100 AD. Descartes was also interested in modelling algebraic operations.

He used the construction on the right to model a quadratic equation.

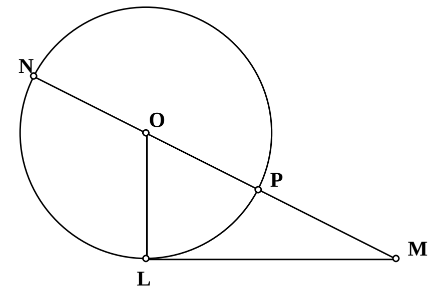


Fig. 8

To see how he did this, you may need to review some geometry of the circle. (see note 2.)

Let $y = MP$, $t = NP$ and $b = LM$. We see that $LM^2 = MP \times MN$, therefore $b^2 = y(y+t)$ or $y^2 + ty - b^2 = 0$, which is a quadratic equation in the variable *y*.

If you let $y = MN$, $t = NP$ and thus, $y - a = PM$. $b = LM$ as before. We see that $LM^2 = MP \times MN$, therefore $b^2 = (y-t)y$ or $y^2 - ty - b^2 = 0$, which is a different form of a quadratic equation in the variable *y*.

The beauty of these constructions is that you can build them with Geometer's Sketchpad (or any other dynamic geometry package) and alter the lengths of *t* and *b* to recreate just about any quadratic equation that you encounter. The *resulting* lengths of *y* will be the *roots* of the equation. You have to realize that Descartes wasn't interested in negative roots as these didn't have much meaning in a geometric sense. In fact he referred to them as "*racines fousses"* or *"false roots" - "less than nothing"*.

Also, there is an added bonus here. Remember that the triangle *LMO* is right-angled (see note 2). This means that, in the first case above we can say that $OM^2 = OL^2 + LM^2$ or $\left(y + \frac{t}{2}\right)^2 = \left(\frac{t}{2}\right)^2 + b^2$. Solving this for *y* gives us: $y = \frac{-t + \sqrt{t^2 + 4b^2}}{2}$. Here we can see the algebraic solution that we are so familiar with.

Repeating this for the second case yields $y = \frac{t + \sqrt{t^2 + 4b^2}}{2}$.

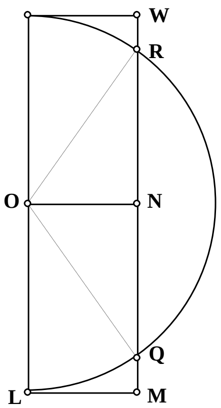


Fig. 3

Descartes was obviously aware that there are other forms of the quadratic equation. The complete list is: $y^2+ty-b^2=0$, $y^2-ty-b^2=0$, $y^2-ty+b^2=0$ and $y^2+ty+b^2=0$. We have dealt with the first two. We will now look at the third and we will ignore the fourth since it cannot have any positive real roots.

For the third case, Descartes used this construction:

If you let $y_1=MQ$, $\frac{t}{2}=OR=OQ=OL$ and thus, $MR=t-y_1$ (note that $WR\cong MQ$). $b=LM$ as before. We see that $LM^2=MQ\times MR$, therefore $b^2=y_1\left(t-y_1\right)$ or $y_1^{\,2}-ty_1+b^2=0$.

We get an algebraic solution by seeing that $y_1=MQ=MN-NQ$. Since $NQ=\sqrt{\left(\frac{t}{2}\right)^2-b^2}$, we end up with $y_1=\frac{t-\sqrt{t^2-4b^2}}{2}$.

If you let $y_2=MR$, $\frac{t}{2}=OR=OQ=OL$ and thus, $MQ=t-y_2$ (note that $WR\cong MQ$). $b=LM$ as before. We see that $LM^2=MQ\times MR$, therefore $b^2=\left(t-y_2\right)y_2$ or $y_2^{\,2}-ty_2+b^2=0$.

We get an algebraic solution by seeing that $y_2=MR=MN+NQ$. Again, since $NQ=\sqrt{\left(\frac{t}{2}\right)^2-b^2}$, we end up with $y_2=\frac{t+\sqrt{t^2-4b^2}}{2}$.

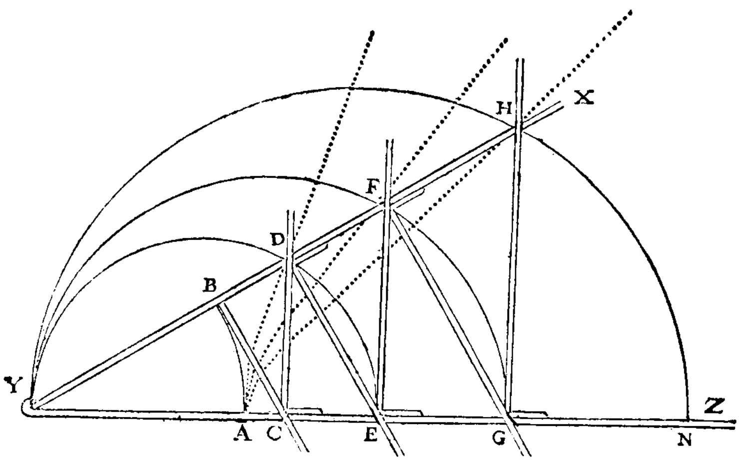
In all, we get the two positive roots that we expected.

To close this section, we should observe that $y_1y_2=b^2$ and $y_1+y_2=t$ - two very familiar results. Also, if $b=\frac{t}{2}$ there will be a single repeated root, while if $b>\frac{t}{2}$, there will be no real roots! The diagram makes these points in a compelling fashion!

(see note 3.)

The second part of Descartes' book is titled *On the Nature of Curved Lines*. Here, he explains how there are many curves worthy of study that were ignored by the ancient Greeks. Using the tools developed in the first part of the book, he proceeds to describe the construction of many curves. The following is a facsimile from his actual text, with an English translation of the pertinent parts.

318 LA GEOMETRIE.



l'ouure, la reigle B C, qui eſt iointe a angles droits auec X Y au point B, pouſſe vers Z la reigle C D, qui coule ſur Y Z en faiſant touſiours des angles droits auec elle, & C D pouſſe D E, qui coule tout de meſme ſur Y X en demeurant parallele a B C, D E pouſſe E F, E F pouſſe F G, cellecy pouſſe G H. & on en peut conceuoir vne infinité d'autres, qui ſe pouſſent conſequutiuement en meſme façon, & dont les vnes facent touſiours les meſmes angles auec Y X, & les autres auec Y Z. Or pendant qu'on ouure ainſi l'angle X Y Z, le point B deſcrit la ligne A B, qui eſt vn cercle, & les autres poins D, F, H, ou ſe font les interſections des autres reigles, deſcriuent d'autres lignes courbes A D, A F, A H, dont les dernieres ſont par ordre plus cõpoſées que la premiere, & cellecy plus que le cercle. mais ie ne voy pas ce qui peut empeſcher, qu'on ne concoiue auſſy nettement, & auſſy diſtinctement la deſcription de cete premiere, que du cercle, ou du

46

Consider the lines AB, AD, AF and so forth, which we may suppose to be described by means of the instrument YZ. This instrument consists of several rulers *hinged* together in such a way that YZ being placed along the line AN the angle XYZ can be *increased* or *decreased* in size, and when its sides are together the points B, C, D, E, F, G, H, all coincide with A; but as the size of the angle is *increased*, the ruler BC, fastened at right angles to XY at the point B, *pushes* toward Z the ruler CD which *slides* along YX always at right angles. In like manner, CD *pushes* DE which *slides* along YX always parallel to BC; DE *pushes* EF; EF *pushes* FG; FG *pushes* GH, and so on. Thus we may imagine an infinity of rulers, each *pushing* another, half of them making equal angles with YX and the rest with YZ.

In using terms such as *hinged*, *increased or decreased in size*, *pushes*, *slides…* it is more than evident that Descartes envisioned some form of dynamic geometry. One has to wonder, if, as he made his sketches, he didn't imagine them coming to life through some sort of seventeenth century animation.

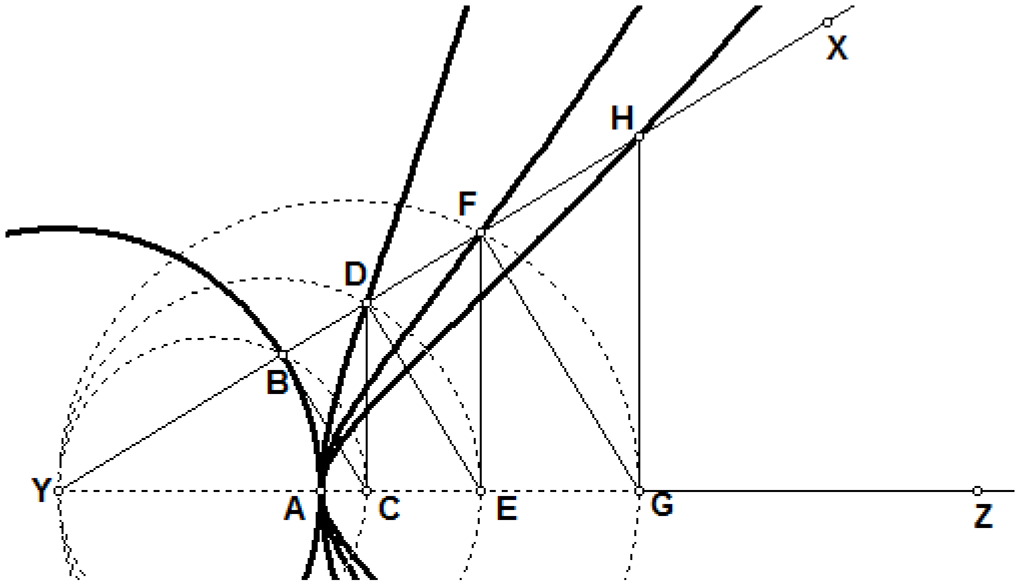


Fig. 15

Using a little proportional reasoning and the Pythagorean theorem he found the equations of the curves drawn here in heavy black. The equations are $x^2 + y^2 = a^2 \left[AB\right]$, $x^4 = a^2\left(x^2 + y^2\right)\left[AD\right]$, $x^8 = a^2\left(x^2 + y^2\right)^3 \left[AF\right]$, and $x^{12} = a^2\left(x^2 + y^2\right)^5 \left[AH\right]$ where *a* is the length YA. There's enough here for at least a semester of school mathematics! In modern terms, if you place the (Cartesian!) origin at the point Y, these equations might appear more "natural". (see note 4.)

Again, I'd like to point out that these equations do not refer to an axis system, but express the inherent relationships amongst the component pieces. However, the transition to the modern "Cartesian" co-ordinate system is immediate as long as we know where to place the origin.

In the third part of the book, *On the Constructions of Solid and Supersolid Problems*, Descartes shows how to find the roots of cubic equations. A *Solid Problem* would be one involving the third dimension!

He first explains how all cubic equations of the form $0 = X^3 + aX^2 + bX + c$ can be re-expressed in the form $0 = x^3 + dx + e$ by substituting $x - \frac{a}{3}$ for *X*. Descartes then explains how each of the various versions of this form can be modelled and "solved" geometrically.

As an example, we will solve the equation $0 = X^3 - 6X^2 + 9X - 4$.

To find the suitable substitution that will remove the quadratic term, we will substitute $x + a$ for X. This gives us: $0 = \left(x+a\right)^3 - 6\left(x+a\right)^2 + 9\left(x+a\right) - 4$.

Expanding and simplifying gives us: $0 = x^3 + 3(a-2)x^2 + 3(a^2 - 4a + 3)x - (6a^2 - 9a + 4)$.

We can now see that if we let $a = 2$, the quadratic term disappears. Thus, we can let $X = x + 2$ and work with the new equation $0 = x^3 - 3x - 2$ or $x^3 = 3x + 2$.

The way Descartes sets about "solving" this equation is quite remarkable. He changes the "appearance" of the equation so that it looks like the solution of a completely different problem - one that can be solved "geometrically".

I'll work with the general case $x^3 = px + q$ in some detail. Bare with me while I develop a scenario. We'll get back to our specific example a little later.

Although Descartes doesn't offer this construction, I found that it helped me to understand what he was saying on page 196 (Dover) of his book!

Consider the circle with centre $\left(\frac{1}{2}q, \frac{1}{2}p + \frac{1}{2}\right)$, passing through the point $(0,0)$. The equation is:

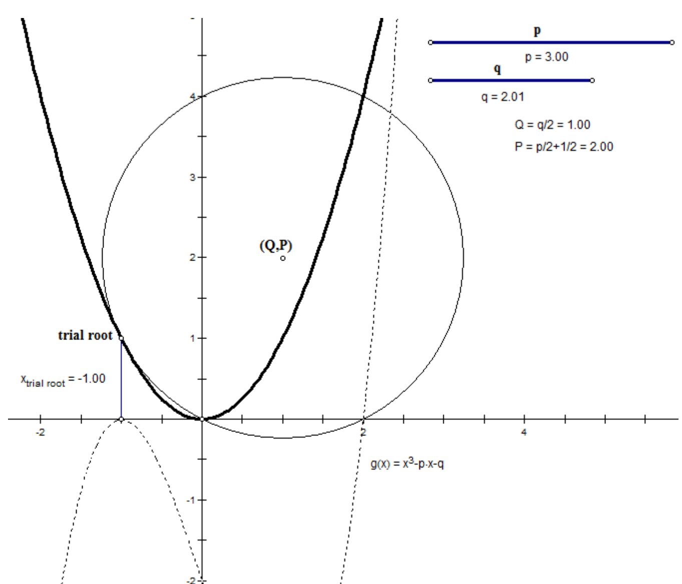
$$\left(x - \frac{1}{2}q\right)^2 + \left(y - \left(\frac{1}{2}p + \frac{1}{2}\right)\right)^2 = \left(\frac{1}{2}q\right)^2 + \left(\frac{1}{2}p + \frac{1}{2}\right)^2.$$

Expanding and simplifying gives us: $x^2 - qx + y^2 - py - y = 0$.

Now, let $y = x^2$, and we get: $x^2 - qx + x^4 - px^2 - x^2 = 0$ which becomes $x^4 = px^2 + qx$ or $x^3 = px + q$ (note: $x = 0$ is an extraneous root).

This means that in order to "construct" the solution to the equation $x^3 = px + q$, all we have to do is construct the circle $\left(x - \frac{1}{2}q\right)^2 + \left(y - \left(\frac{1}{2}p + \frac{1}{2}\right)\right)^2 = \left(\frac{1}{2}q\right)^2 + \left(\frac{1}{2}p + \frac{1}{2}\right)^2$ and the parabola $y = x^2$ and look for the points of intersection.

In the accompanying diagram, you can see the circle and parabola. I have included part of the curve $y = x^3 - 3x - 2$ so that you can see that the intersections of the parabola and circle do in fact correspond to the roots. This is a particularly non-Cartesian construction as Descartes would have had it, but this sort of hybrid is probably more familiar nowadays. In the GSP sketch, as you move the "trial root" point around the circle you can see how the roots and points of intersection correspond. You can also see the "double root" $x = -1$ ... even if Descartes ignored it. By adjusting the lengths of segments ***p*** and ***q*** you can change the original equation. You can make these values negative and so cover all eventualities.



Don't forget, to find the roots of the original equation - the one before the transformation - you will need to translate the roots accordingly.

When Descartes performs the algebraic transformation of $X = x - \frac{a}{3}$ in the equation $0 = X^3 + aX^2 + bX + c$, in order to obtain $0 = x^3 + dx + e$, he is actually performing a geometric transformation as well. By transforming the function $y = x^3 + ax^2 + bx + c$, he is simply translating the curve so that the inflexion point ends up sitting on the ***y*** axis. We can prove this by looking through the lens of calculus ... unfortunately, Descartes probably wasn't aware of this since the calculus wouldn't be invented for another 30 years or so. All the same it's a pretty slick move!

Descartes offers some words of encouragement to his readers and some pretty sound advice.

"I have omitted here the demonstration of most of my statements, because they seem to me so easy that if you take the trouble to examine them systematically the demonstrations will present themselves to you and it will be much more valuable to you to learn them in that way than by reading them."

I can just imagine his response to my complaints about my difficulties in reading his text.

"Mais, il est évident."

notes:

1. Given $1)\, y = x^2$ and $2)\, ax = y^2$, let's find the point of intersection. 1) gives us $3)\, y^2 = x^4$. 2) and 3) give us $x^4 = ax$. Thus $x^3 = a$ and $x = \sqrt[3]{a}$ (as long as $x \neq 0$). Here, I'm again using a hybrid of methods.

2. First, consider the following diagram:

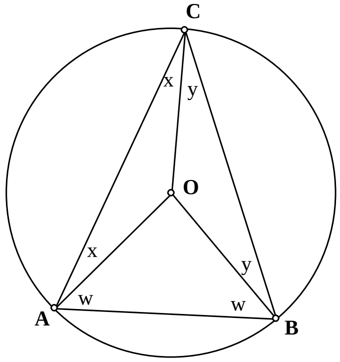


Fig. 9

What is the relationship between $\angle ACB$ and $\angle AOB$? Does this change as the point *C* moves around the circle? What happens if the point *C* is placed on the smaller arc *AB*?

This result is summarised by saying that angles "subtended" by the same chord are congruent.

Now, consider this diagram:

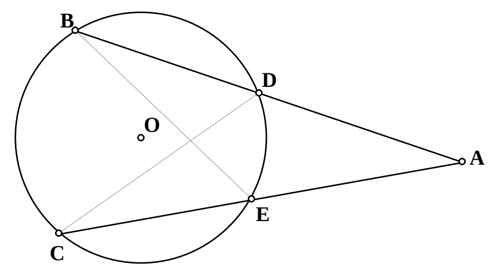


Fig. 10

Since $\angle DBE \cong \angle DCE$, $\triangle ABE \sim \triangle ACD$ (two angles are congruent).

Therefore: $\dfrac{AB}{AC} = \dfrac{AE}{AD} = \dfrac{BE}{CD}$, in particular: $AB \times AD = AC \times AE$.

If the points *A* and *C* are allowed to moved so that *AC* is actually tangent - *C* and *E* will become a single point - then the equation $AB \times AD = AC \times AE$ becomes $AD \times AB = AC^2$.

If you move the point B so that BA passes through the centre, you will get the diagram on the right.

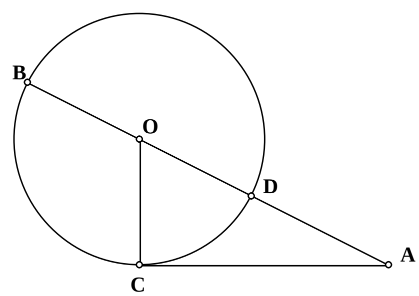


Fig. 11

Since we know that:

$$
\begin{aligned}
AC^2 &= AD \times AB \\
&= (AO - OC)(AO + OC) \\
&= AO^2 - OC^2 \\
\therefore AO^2 &= AC^2 + OC^2 \\
\therefore \angle OCA &= 90^\circ
\end{aligned}
$$

Thus *OC* is perpendicular to *CA*.

3. Here is a model that has been developed more recently.

As you can see, the diagram has been drawn on the Cartesian plane. *P* is a variable point on the x-axis and *Q* is a variable point on the y-axis. *C* is at (0,1) and *O* is the mid-point of *CF*.

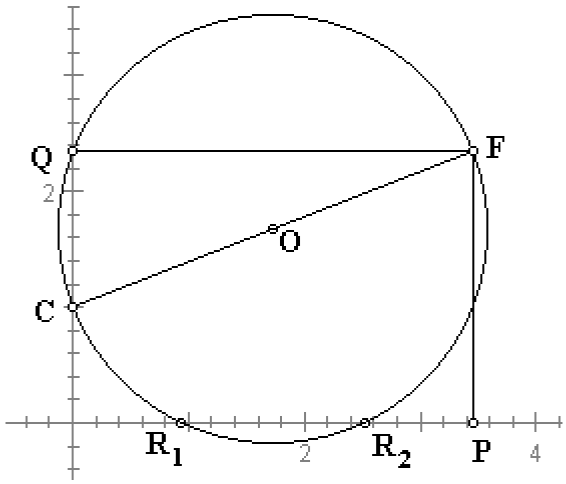


Fig. 14

To understand how this model works you will first need to understand the significance of R_1 and R_2.

Let the points P, Q, R_1 and R_2 be (p, 0), (0, q), (z_1,0), and (z_2,0).

a) Explain why $p = z_1 + z_2$.

b) Explain why $q = z_1 \times z_2$.

c) Show that $z_1 = \dfrac{p - \sqrt{p^2 - 4q}}{2}$ and $z_2 = \dfrac{p + \sqrt{p^2 - 4q}}{2}$ geometrically.

d) Explain how this construction models *all* quadratic equations - even ones with negative roots.

d) Using the Geometer's Sketchpad, construct a "quadratic equation solver" and solve the following equations:

i) $x^2 - 5x + 6 = 0$ ii) $x^2 + x - 6 = 0$ iii) $x^2 + 5x + 5 = 0$

4. Consider AD. Let $YA = YB = a$, $YC = x$, $CD = y$ and $YD = z$. By similar triangles, □YBC □□YCD, we have $YD : YC = YC : YB$ or $z : x = x : a$. Thus $z = \dfrac{x^2}{a}$ or $z^2 = \dfrac{x^4}{a^2}$. But $z^2 = y^2 + x^2$. Leaving us with $x^4 = a^2\left(x^2 + y^2\right)$. The other equations pretty well follow this pattern.